Find:

$$\lim_{x\to 0}\frac{1-\cos(x)\cdot\sqrt{\cos(2x)}\cdot\sqrt[3]{\cos(3x)}\cdot\ldots\cdot\sqrt[n]{\cos(nx)}}{x^2}$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$L=\lim_{x\to 0}\frac{1-\cos(x)\cdot\sqrt{\cos(2x)}\cdot\sqrt[3]{\cos(3x)}\cdot\ldots\cdot\sqrt[n]{\cos(nx)}}{x^2}=\lim_{x\to 0}\frac{1-\prod_{k=1}^{n}\left(\cos(kx)\right)^{\frac{1}{k}}}{x^2}$$

$$L=\lim_{x\to 0}\frac{1-f(nx)}{x^2}\to f(nx)=\prod_{k=1}^{n}(\cos(kx))^{\frac{1}{k}}$$

Using the Maclaurin series expansion for $\cos(kx)$ *as* $x\to 0$:

$$\cos(kx)=\sum_{m=0}^{\infty}\frac{(-1)^m(kx)^{2m}}{(2m)!}\to\cos(\mathrm{kx})\sim 1-\frac{(kx)^2}{2}\ \ as\ x\to 0$$

The expression is evaluated by applying the generalized binomial expansion for |x| <<1. Specifically, we use the first-order linear approximation derived from Newton's binomial theorem: $(1+x\beta)^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}\beta x^k\sim 1-\alpha\beta x,\ \ as\ \ x\to 0$

$$f(nx)=\prod_{k=1}^{n}(\cos(kx))^{\frac{1}{k}}\sim\prod_{k=1}^{n}\left(1-\frac{(kx)^2}{2}\right)^{\frac{1}{k}}\sim\prod_{k=1}^{n}\left(1-\frac{kx^2}{2}\right)$$

$$f(nx)\sim\prod_{k=1}^{n}\left(1-\frac{kx^2}{2}\right)=\left(1-\frac{x^2}{2}\right)(1-x^2)\left(1-\frac{3x^2}{2}\right)\ldots\left(1-\frac{nx^2}{2}\right)$$

Weierstrass Product Inequality: $\prod_{i=1}^{n}(1-a_i)\geq 1-\sum_{k=1}^{n}a_i$

$$f(nx)\sim\prod_{k=1}^{n}\left(1-\frac{kx^2}{2}\right)\sim 1-\sum_{k=1}^{n}\frac{kx^2}{2}=1-\frac{x^2}{2}\sum_{k=1}^{n}k=1-\frac{x^2n(n+1)}{4}$$

$$L=\lim_{x\to 0}\frac{1-f(nx)}{x^2}=\lim_{x\to 0}\frac{1-\left(1-\frac{x^2n(n+1)}{4}\right)}{x^2}=\frac{n(n+1)}{4}$$

$$\lim_{x\to 0}\frac{1-\cos(x)\cdot\sqrt{\cos(2x)}\cdot\sqrt[3]{\cos(3x)}\cdot\ldots\cdot\sqrt[n]{\cos(nx)}}{x^2}=\frac{n(n+1)}{4}$$