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Prove that:

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\int_{\frac{k}{n}}^{\frac{k}{n}+1}\ln\left(\Gamma(t)\right)dt\right)=-\frac{3}{4}+\frac{1}{2}\ln(2\pi)$$

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$$\Omega=\lim_{n\to+\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\int_{\frac{k}{n}}^{\frac{k}{n}+1}\ln\big(\Gamma(t)\big)\,dt\right)=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)$$

$$\textit{Limit or Riemann sums}:\ \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)=\int_0^1 f(x)dx$$

$$\Omega=\int_0^1\int_x^{1+x}\ln\big(\Gamma(t)\big)\,dt\,dx=\int_0^1 f(x)\,dx$$

$$\textit{Raabe's integral}:\ f(x)=\int_x^{1+x}\ln\Gamma(t)dt$$

$$\frac{d}{dx}f(x)=\ln\Gamma(1+x)\frac{d}{dx}(1+x)-\ln\Gamma(x)\frac{d}{dx}x=\ln\Gamma(1+x)-\ln\Gamma(x)$$

$$\Gamma(1+x)=x\Gamma(x)\to\frac{d}{dx}f(x)=\ln(x)\to\int\ln(x)\,dx=x\ln(x)-x+C$$

$$C=f(0)=\int_0^1\ln\Gamma(t)\,dt\ \underline{\overset{(1-t)\to t}{\quad}}\ C=\int_0^1\ln\Gamma(1-t)dt$$

$$2C=\int_0^1\ln\big(\Gamma(1-t)\Gamma(t)\big)\,dt$$

$$\textit{Gamma reflection formula}:\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin(\pi z)},\qquad z\notin Z$$

$$2C=\int_0^1\ln(\pi)\,dt-\int_0^1\ln\big(\sin(\pi t)\big)\,dt=\ln(\pi)-\underbrace{\int_0^1\ln\big(\sin(\pi t)\big)\,dt}_{\frac{\pi t}{2}=u}=\ln(\pi)-K$$

$$K=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin(2u)\,du=\frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\ln(2)\,du+\int_0^{\frac{\pi}{2}}\ln\big(\sin(u)\big)\,du+\int_0^{\frac{\pi}{2}}\ln\big(\cos(u)\big)\,du\right)$$

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$$simmetry \rightarrow \underbrace{\int_0^{\frac{\pi}{2}} \ln(\sin(u))\, du}_{\frac{\pi}{2}-u\rightarrow u} = \int_0^{\frac{\pi}{2}} \ln(\cos(u))\, du = \frac{1}{2}\int_0^{\pi} \ln(\sin(u))\, du$$

$$K = \frac{2}{\pi}\left(\frac{\pi}{2}\ln(2) + \underbrace{\int_0^{\pi} \ln(\, sin(u))\, du}_{u\pi\rightarrow u}\right) = \frac{2}{\pi}\left(\frac{\pi}{2}\ln(2) + \pi K\right) \rightarrow K = -\ln(2)$$

$$2C = \ln(2) - K \rightarrow\ C = \frac{1}{2}\ln(2\pi) \quad f(x) = x\ln(x) - x + \frac{1}{2}\ln(2\pi)$$

$$\Omega = \int_0^1 f(x)dx = \underbrace{\int_0^1 x\ln(x)\, dx}_{IBP} - \left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\ln(2\pi) = \left[\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right]_0^1 - \frac{1}{2} + \frac{1}{2}\ln(2\pi)$$

$$= -\frac{1}{4} - \frac{1}{2} + \frac{1}{2}\ln(2\pi) = -\frac{3}{4} + \frac{1}{2}\ln(2\pi)$$

$$Therefore\ \ \lim_{n\rightarrow+\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\int_{\frac{k}{n}}^{\frac{k}{n}+1} \ln\Gamma(t)dt\right) = -\frac{3}{4} + \frac{1}{2}\ln(2\pi)$$