ROMANIAN MATHEMATICAL MAGAZINE

$$u_n=\frac{n^a\left(\prod_{k=1}^{n}\pi k^k\right)^{-\frac{2a}{n^2}}}{\pi\sum_{k=0}^{n}\frac{1}{k+n}},\quad v_n=\frac{n^a\left(\prod_{k=1}^{n}\pi k^k\right)^{-\frac{2a}{n^2}}}{\pi\sum_{k=n}^{2n}\sin\left(\frac{1}{k}\right)},\quad a>0$$

$$Find\ \lim_{n\to+\infty}(u_n)\ \ and\ \lim_{n\to+\infty}(v_n)$$

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$$u_n=\frac{n^a\left(\prod_{k=1}^{n}\pi k^k\right)^{-\frac{2a}{n^2}}}{\pi\sum_{k=0}^{n}\frac{1}{k+n}}=\frac{S_n}{K_n}\ \to\ \begin{cases} S_n=n^a\left(\prod_{k=1}^{n}\pi k^k\right)^{-\frac{2a}{n^2}} \\ K_n=\pi\sum_{k=0}^{n}\frac{1}{k+n}\end{cases}$$

$$\ln(S_n)=\ln\left(n^a\left(\prod_{k=1}^{n}\pi k^k\right)^{-\frac{2a}{n^2}}\right)=a\ln(n)-\frac{2a}{n^2}\ln\left(\prod_{k=1}^{n}\pi k^k\right)=$$

$$=a\ln(n)-\frac{2a}{n^2}\sum_{k=1}^{n}\ln\left(\pi k^k\right)=a\ln(n)-\frac{2a}{n^2}\left(n\ln(\pi)+\sum_{k=1}^{n}k\ln(k)\right)=$$

$$=a\ln(n)-\frac{2a}{n}\ln(\pi)-\frac{2a}{n^2}L_n$$

f(x)=x ln(x) is increasing on the interval $[1;\infty)$

$$\int_0^n f(x)dx\le\sum_{k=1}^{n}f(k)\le\int_1^{n+1}f(x)dx\ \to\ f(x)=x\ln(x)$$

$$\int_0^n x\ln(x)\,dx\le\sum_{k=1}^{n}k\ln(k)\le\int_1^{n+1}x\ln(x)\,dx$$

$$\left[\frac{x^2}{2}\ln(x)-\frac{x^2}{4}\right]_0^n\le\sum_{k=1}^{n}k\ln(k)\le\left[\frac{x^2}{2}\ln(x)-\frac{x^2}{4}\right]_1^{n+1}$$

$$\frac{n^2}{2}\ln(n)-\frac{n^2}{4}\le\sum_{k=1}^{n}k\ln(k)\le\frac{(n+1)^2\ln(n+1)}{2}-\frac{n^2}{4}-\frac{n}{2}$$

$$L_n=\sum_{k=1}^{n}k\ln(k)\approx\frac{n^2}{2}\ln(n)-\frac{n^2}{4}+O(n\ln(n))$$

$$\ln S_n = a\ln(n) - \frac{2a}{n}\ln(\pi) - \frac{2a}{n^2}\left(\frac{n^2}{2}\ln(n) - \frac{n^2}{4}\dots\right) = -\frac{2a}{n}\ln(\pi) + \frac{a}{2}$$

$$S_n = \frac{e^{\frac{a}{2}}}{e^{\frac{2a}{n}\ln(\pi)}} = \frac{e^{\frac{a}{2}}}{\pi^{\frac{2a}{n}}} \to \lim_{n\to+\infty} S_n = \frac{e^{\frac{a}{2}}}{\pi^0} = e^{\frac{a}{2}}$$

$$\to K_n = \pi\sum_{k=0}^{n}\frac{1}{n+k} = \frac{\pi}{n}\sum_{k=0}^{n}\frac{1}{\left(1+\frac{k}{n}\right)}$$

$$\textit{Riemann Sum Approximation}:\ \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx$$

$$\lim_{n\to+\infty} K_n = \pi\int_0^1 \frac{1}{1+x}dx = \pi\ln(2),\qquad \lim_{n\to+\infty}(u_n) = \frac{\lim_{n\to+\infty}(S_n)}{\lim_{n\to+\infty}(K_n)} = \frac{e^{\frac{a}{2}}}{\pi\ln(2)}$$

$$v_n = \frac{n^a\left(\prod_{k=1}^{n}\pi k^k\right)^{-\frac{2a}{n^2}}}{\pi\sum_{k=n}^{2n}\sin\left(\frac{1}{k}\right)} = \frac{S_n}{M_n} \to M_n = \pi\sum_{k=n}^{2n}\sin\left(\frac{1}{k}\right)$$

$$\textit{Taylor series}:\ \sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin\left(\frac{1}{k}\right) = \frac{1}{k} - \frac{1}{k^3 3!} + \frac{1}{k^5 5!} - \cdots = \frac{1}{k} - \frac{1}{6k^3} + \frac{1}{120k^5} + O\left(\frac{1}{k^6}\right)$$

$$\sum_{k=n}^{2n}\sin\left(\frac{1}{k}\right) \sim \sum_{k=n}^{2n}\frac{1}{k} = \sum_{k=1}^{2n}\frac{1}{k} - \sum_{k=1}^{n-1}\frac{1}{k} = H_{2n} - H_{n-1}$$

Using the asymptotic expansion of harmonic numbers

$$H_n = \ln(n) + \gamma + O\left(\frac{1}{n}\right) \to \begin{cases} H_{n-1} = H_n - \frac{1}{n} = \ln(n) + \gamma - 1/n + O(1/n) \\ H_{2n} = \ln(2n) + \gamma + O\left(\frac{1}{2n}\right) \end{cases}.$$

$$M_n = \pi(H_{2n} - H_{n-1}) = \pi\left(\ln(2n) - \ln(n) - \frac{1}{n} + O\left(\frac{1}{n}\right) - O\left(\frac{1}{2n}\right)\right)$$

$$\lim_{n\to+\infty} M_n = \pi\ln(2) \to \lim_{n\to+\infty}(v_n) = \frac{\lim_{n\to+\infty} S_n}{\lim_{n\to+\infty} M_n} = \frac{e^{\frac{a}{2}}}{\pi\ln(2)}$$

$$\textit{Therefore}\quad \lim_{n\to+\infty}(u_n) = \lim_{n\to+\infty}(v_n) = \frac{e^{\frac{a}{2}}}{\pi\ln(2)}$$