ROMANIAN MATHEMATICAL MAGAZINE

Find :

$$\Omega = \lim_{n\to\infty} \frac{n+2}{2^n} \sum_{k=0}^{n} \frac{(-1)^k . 2^{n-k}}{k+1} . \binom{n}{k}$$

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Solution by Shirvan Tahirov-Azerbaijan

$$\Omega = \lim_{n\to\infty} \frac{n+2}{2^n} \sum_{k=0}^{n} \frac{(-1)^k . 2^{n-k}}{k+1} . \binom{n}{k} = \lim_{n\to\infty} \frac{n+2}{2^n} \sum_{k=0}^{n} \frac{(-1)^k . 2^n . 2^{-k}}{k+1} . \binom{n}{k} =$$

$$= \lim_{n\to\infty} \frac{n+2}{2^n} . 2^n \sum_{k=0}^{n} \frac{(-1)^k}{2^k (k+1)} . \binom{n}{k} = \lim_{n\to\infty} (n+2) \sum_{k=0}^{n} (-\frac{1}{2})^k \binom{n}{k} \int_0^1 x^k dx =$$

$$= \lim_{n\to\infty} (n+2) \int_0^1 \sum_{k=0}^{n} (-\frac{x}{2})^k \binom{n}{k} dx = \lim_{n\to\infty} (n+2) \int_0^1 \left(1-\frac{x}{2}\right)^n dx \overset{\begin{cases} t=1-\frac{x}{2} \\ dx=-2dt \end{cases}}{=}$$

$$= \lim_{n\to\infty} (n+2) \left(\int_{\frac{1}{2}}^{1} 2t^n dt \right) = \lim_{n\to\infty} (n+2) \left(\frac{2}{n+1} . t^{n+1} \Big|_{\frac{1}{2}}^{1} \right) =$$

$$= \lim_{n\to\infty} (n+2) \left(\frac{2-2^{-n}}{n+1} \right) = \lim_{n\to\infty} \left(\frac{n+2}{n+1} . (2-2^{-n}) \right) = \lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) = 2$$