ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}{1!\sqrt{2!\sqrt{3!\cdot\ldots\cdot\sqrt{n!}}}}\right)^n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Hikmat Mammadov-Azerbaijan

$$A_n=1!+\sqrt{2!+\sqrt{3!+\sqrt{n!}}},\quad B_n=1!\sqrt{2!\sqrt{3!\ldots\sqrt{n!}}}$$

$$\frac{A_n}{B_n}=\frac{1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}{1!\sqrt{2!\sqrt{3!\ldots\sqrt{n!}}}}=\frac{1!}{B_n}+\frac{\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}{1!\sqrt{2!\sqrt{3!\ldots\sqrt{n!}}}}$$

Note: $1!\cdot 2!\cdot\sqrt{\sqrt{3!\ldots}}=1!\cdot\sqrt{2!}\cdot B'_n$, $\frac{A_n}{B_n}=\frac{1}{B_n}+\frac{1}{1!}\cdot\sqrt{\frac{1}{2!}+\frac{\sqrt{3!+\cdots}}{2!\cdot\sqrt{3!\ldots}}}$

$$n\to\infty\Rightarrow 1<\frac{A_n}{B_n}<1+\frac{C}{2^n}\quad (C=\text{constant})1^n<\left(\frac{A_n}{B_n}\right)^n<\left(1+\frac{C}{2^n}\right)^n$$

$$\lim_{n\to\infty}\left(1+\frac{C}{2^n}\right)^n=\lim_{n\to\infty}e^{\frac{n\cdot C}{2^n}}=e^0=1\Rightarrow\Omega=1$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$A_n=1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}},\quad B_n=1!\sqrt{2!\sqrt{3!\ldots\sqrt{n!}}}=\prod_{k=1}^{n}(k!)^{\frac{1}{2^{k-1}}}$$

$$\frac{A_n}{B_n}=\frac{1!+\sqrt{2!+\sqrt{3!+\cdots+\sqrt{n!}}}}{\prod_{k=1}^{n}(k!)^{\frac{1}{2^{k-1}}}}=1+O\left(\frac{1}{2^n}\right),$$

$$\Omega=\lim_{n\to\infty}\left(1+\left(\frac{A_n}{B_n}-1\right)\right)^n=e^{\lim\limits_{n\to\infty}n\cdot\left(\frac{A_n}{B_n}-1\right)}$$

$$\lim_{n\to\infty}n\cdot O\left(\frac{1}{2^n}\right)=\lim_{n\to\infty}\frac{n}{2^n}=0$$

$$\Omega=e^0=1\Rightarrow\Omega=1$$