ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right),\ where\ \lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}=a\in R_+^*$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$Stirling\ formula\quad n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$(2n-1)!!=\frac{(2n)!}{2^n n!}\sim\left(\frac{2n}{e}\right)^n\to\sqrt[n]{(2n-1)!!}\sim\frac{2n}{e}$$

Cauchy-D'Alembert criterion: $A=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\sqrt[n]{n!}$

$If\ \frac{a_{n+1}}{a_n}\sim n^k$ This implies that the sequence a_n grows at an asymptotic

. rate of $(n!)^k$. $\frac{a_{n+1}}{a_n}\sim Cn^k\to a_n=a_1\cdot\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdot\ldots\cdot\frac{a_{n+1}}{a_n}$

$$a_n\sim a_1(C\cdot 1^k)(C\cdot 2^k)\ldots(C\cdot n^k)=C^n(n!)^k\sim C^n\left(\frac{n}{e}\right)^{nk}$$

$$Asyptotik\ approach\quad \sqrt[n]{a_n}\sim C\left(\frac{n}{e}\right)^k$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}=a\to\frac{a_{n+1}}{a_n}\sim a\sqrt[n]{(2n-1)!!}\sim\frac{2an}{e}$$

$$C=\frac{2a}{e},k=1\to\sqrt[n]{n!}\sim\frac{2a}{e}\cdot\frac{n}{e}=\frac{2an}{e^2}$$

$$L=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)=\lim_{n\to\infty}\left(\frac{(n+1)^2}{\frac{2a(n+1)}{e^2}}-\frac{n^2}{\frac{2an}{e^2}}\right)=$$

$$=\frac{e^2}{2a}\lim_{n\to\infty}((n+1)-n))=\frac{e^2}{2a}$$