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$$\textit{Find} \lim_{n\to\infty}\left(\frac{\pi^2}{4}-a_n^2\right)n\,,\textit{where } a_n=\sum_{k=1}^{n}\arctan\frac{1}{k^2-k+1}$$

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Solution by Tapas Das-India

$$\arctan\frac{1}{k^2-k+1}=\arctan\frac{k-(k-1)}{1+k(k-1)}=\arctan(k)-\arctan(k-1)$$

$$\textit{then,}\sum_{k=1}^{n}\arctan\frac{1}{k^2-k+1}$$

$$=(\arctan 1-\arctan 0)+(\arctan 2-\arctan 1)+\cdots+(\arctan(n)-\arctan(n-1))$$

$$=\arctan(n)-\ \arctan(0)=\arctan(n)$$

$$\lim_{n\to\infty}\left(\frac{\pi^2}{4}-a_n^2\right)n=\lim_{n\to\infty}\left(\frac{\pi^2}{4}-(\arctan(n))^2\right)n=\lim_{z\to 0}\frac{\left(\frac{\pi^2}{4}-\left(\arctan\left(\frac{1}{z}\right)\right)^2\right)}{z}$$

$$\left(\textit{let } n=\frac{1}{z},\textit{when } n\to\infty \textit{ then } z\to 0\right)$$

$$\textit{using } L'\textit{Hospital Rule}\left(\frac{0}{0}\right)\textit{from we get}$$

$$\lim_{z\to 0}\frac{\left(-2\arctan\frac{1}{z}\right).\frac{z^2}{z^2+1}.\left(-\frac{1}{z^2}\right)}{1}=\lim_{z\to 0}\frac{\left(2\arctan\frac{1}{z}\right).\frac{1}{z^2+1}.}{1}.=2\times\frac{\pi}{2}=\pi$$