$$Find \ \lim_{n\to\infty} \frac{\sqrt[n]{x_n}}{n}, where \ x_n = \prod_{k=1}^{n} a_k^{\frac{1}{k}} \ and \ (a_n)_{n\geq 1} > 0$$

$$such \ that \ \lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{n!}} = a \in R_+^*$$

**Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania**

*Solution by Amin Hajiyev-Azerbaijan*

The limit form of Stirling's Formula: $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = \lim_{n\to\infty} \left( \frac{a_{n+1}}{a_n \sqrt[n]{n!}} \cdot \frac{\sqrt[n]{n!}}{n} \right) = \frac{a}{e}$$

Cauchy-D'Alembert criterion: $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = y \to \lim_{n\to\infty} \sqrt[n]{b_n} = y$

$$b_n = \frac{a_n}{(n-1)!} \to \frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{\frac{n!}{(n-1)!} a_n} = \frac{a_{n+1}}{na_n}$$

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} \sqrt[n]{\left( \frac{a_n}{(n-1)!} \right)} = \frac{a}{e}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\left( \frac{a_n}{(n-1)!} \right)} \cdot \frac{\sqrt[n]{(n-1)!}}{n} = \frac{a}{e} \cdot \frac{1}{e} = \frac{a}{e^2}$$

$$x_n = \prod_{k=1}^{n} a_k^{\frac{1}{k}} \to x_{n+1} = x_n \cdot \sqrt[n+1]{a_{n+1}}$$

$$L = \lim_{n\to\infty} \frac{\sqrt[n]{x_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{x_n}{n^n}} \overset{Cauchy-D'Almbert}{\hat{=}} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x_n} =$$

$$= \lim_{n\to\infty} \sqrt[n+1]{\frac{a_{n+1}}{(n+1)^{n+1}}} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} \cdot \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{a}{e^2} \cdot \frac{1}{e} = \frac{a}{e^3}$$

$$Therefore \ \lim_{n\to\infty} \frac{\sqrt[n]{x_n}}{n} = \frac{a}{e^3}$$