ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

$$where\ (a_n)_{n\geq 1}>0,\ such\ that\ \lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}=a\in R_+^*$$

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Solution by Amin Hajiyev-Azerbaijan

$$(2n-1)!!=\frac{(2n)!}{2^n n!},\qquad Stirling\ formula,\qquad n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\left((2n-1)!!\right)^{\frac{1}{n}}\sim\frac{2n}{e}\to\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\frac{2}{e}$$

- **Stolz-Cesàro theorem:** $(x_n)_{n\geq 1},(y_n)_{n\geq 1}\to\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)=\lim_{n\to\infty}\left(\frac{x_{n+1}-x_n}{y_{n+1}-y_n}\right)=M$

$$S=\lim_{n\to\infty}\sqrt[n]{a_n}\,,\ln S=\lim_{n\to\infty}\left(\frac{\ln(a_n)}{n}\right)=\lim_{n\to\infty}\frac{\ln(a_{n+1})-\ln(a_n)}{n+1-n}$$

$$S=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$$

- **Cauchy-D'Alembert criterion:** $\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\sqrt[n]{b_n}=K$

$$b_n=\frac{a_n}{n^n}\to L=\lim_{n\to\infty}\sqrt[n]{b_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{a_n}{n^n}\right)}=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\frac{\frac{a_{n+1}}{(n+1)^{n+1}}}{\frac{a_n}{n^n}}$$

$$=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\cdot\frac{1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n\right)$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{(2n-1)!!}}=a\to\ Asymptotik\ approach\frac{a_{n+1}}{a_n}\sim a\sqrt[n]{(2n-1)!!}$$

$$L=\lim_{n\to\infty}a\sqrt[n]{(2n-1)!!}\cdot\frac{1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n=\frac{2a}{e}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1}=$$

$$=\frac{2a}{e}\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}=\frac{2a}{e}\cdot\frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}}=\frac{2a}{e^2}$$