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Find:

$$\lim_{n\to\infty} n^{1-a}\cdot\sqrt[n]{\frac{\left((3n)!!\right)^a}{(3an)!!}},\qquad where\ a\in N^*$$

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According to the Ratio-to-Root limit theorem (Cauchy-D'Alembert criterion):

$$If\ \lim_{n\to\infty}\frac{x_{n+1}}{x_n}=L,\qquad then\qquad \lim_{n\to\infty}\sqrt[n]{x_n}=L$$

$$Where\ L\in[0;\infty),\qquad \{x_n\}_{n\in N}$$

Stirling formula: $n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \sqrt[n]{(2n)!!}=\sqrt[n]{2^n n!}\sim\left(\frac{2n}{e}\right)\ as\ n\to\infty$

$$L=\lim_{n\to\infty} n^{1-a}\cdot\sqrt[n]{\frac{\left((3n)!!\right)^a}{(3an)!!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^{n(1-a)}\left((3n)!!\right)^a}{(3an)!!}}=\lim_{n\to\infty}\sqrt[n]{x_n}$$

$$L=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{\dfrac{(n+1)^{(n+1)(1-a)}\left(\left(3(n+1)\right)!!\right)^a}{\left(3a(n+1)\right)!!}}{\dfrac{n^{n(1-a)}\left((3n)!!\right)^a}{(3an)!!}}$$

$$=\lim_{n\to\infty}\frac{(n+1)^{(n+1)(1-a)}\left(\left(3(n+1)\right)!!\right)^a(3an)!!}{n^{n(1-a)}\left(3a(n+1)\right)!!\left((3n)!!\right)^a}$$

Derivation of the Asymptotic Relation: $(2n)!!=2^n n!\to\frac{(2n+2m)!!}{(2n)!!}=\frac{2^m(m+n)!}{n!}$

$$\frac{(n+m)!}{n!}=(n+1)(n+2)\dots(n+m)\sim n^m,\qquad n=\frac{c}{2},m=\frac{b}{2}$$

$$\frac{(c+b)!!}{c!!}\sim 2^{\frac{b}{2}}\left(\frac{c}{2}\right)^{\frac{b}{2}}=\frac{2^{\frac{b}{2}}c^{\frac{b}{2}}}{2^{\frac{b}{2}}}=c^{\frac{b}{2}}$$

$$\frac{\left((3n+3)!!\right)^a}{\left((3n)!!\right)^a}\sim(3n)^{\frac{3a}{2}},\qquad \frac{(3an)!!}{(3an+3a)!!}\sim\frac{1}{(3an)^{\frac{3a}{2}}}$$

$$L=\lim_{n\to\infty}\frac{(n+1)^{(n+1)(1-a)}n^{an}(3n)^{\frac{3a}{2}}}{n^{n(1-a)}(3n)^{\frac{3a}{2}}a^{\frac{3a}{2}}}=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{n}\right)^{1-a}(1+n)^{1-a}a^{-\frac{3a}{2}}$$

Using the fundamental limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$, **we obtain the following asymptotic behavior:** $\left(\left(1+\frac{1}{n}\right)^{n}\right)^{1-a}(1+n)^{1-a}\sim e^{1-a}(1+n)^{1-a}\ \ as\ \ n\to\infty$

$$L=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{n}\right)^{1-a}(1+n)^{1-a}a^{-\frac{3a}{2}}=\lim_{n\to\infty}e^{1-a}(1+n)^{1-a}a^{-\frac{3a}{2}}$$

$$L=e^{1-a}a^{-\frac{3a}{2}}\lim_{n\to\infty}\frac{1+n}{(1+n)^{a}}=0\ \ as\ \forall a\in N,a>1$$