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If $(H_n)_{n\geq 1}, H_n = \sum_{k=1}^{n} \frac{1}{k} \to$ *(Harmonik numbers) then prove*

$$\lim_{n\to\infty} e^{-H_n} \sum_{k=1}^{n} \frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$I = \lim_{n\to\infty} e^{-H_n} \sum_{k=1}^{n} \frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}} = \lim_{n\to\infty} \frac{1}{e^{H_n}} \sum_{k=1}^{n} \frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}} = \lim_{n\to\infty} \frac{a_n}{b_n}$$

By the Stolz-Cesàro theorem: $I \overset{C-S}{\equiv} \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$

$$a_{n+1} - a_n = \sum_{k=1}^{n+1} \frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}} - \sum_{k=1}^{n} \frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}} = \frac{e^{H_{n+1}}}{\sqrt[n+1]{(2n+1)!!}}$$

$$b_{n+1} - b_n = e^{H_{n+1}} - e^{H_n} \overset{H_n = H_{n+1} - \frac{1}{n+1}}{\equiv} e^{H_n}\left(e^{\frac{1}{n+1}} - 1\right)$$

$$I = \lim_{n\to\infty} \frac{e^{H_{n+1}}}{e^{H_n}\left(e^{\frac{1}{n+1}} - 1\right)\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \frac{e^{H_{n+1}-H_n}}{\left(e^{\frac{1}{n+1}} - 1\right)\sqrt[n+1]{(2n+1)!!}}$$

Stirling formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad (2n-1)!! = \frac{(2n)!}{2^n n!} \sim \frac{2\sqrt{\pi n}\left(\frac{2n}{e}\right)^{2n}}{2^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = \sqrt{2}\left(\frac{2n}{e}\right)^n$$

$$(2n+1)!! \sim \sqrt{2}\left(\frac{2(n+1)}{e}\right)^{n+1}$$

$$e^x - 1 \sim x \text{ as } x \to 0 \qquad e^{\frac{1}{n+1}} - 1 \sim \frac{1}{n+1} \to \lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$$

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$$I=\lim_{n\to\infty}\frac{e^{\frac{1}{n+1}}}{\frac{\left(e^{\frac{1}{n+1}}-1\right)2^{\frac{1}{2(n+1)}}2(n+1)}{e}}=\frac{e}{2}\frac{\lim\limits_{n\to\infty}e^{\frac{1}{n+1}}}{\lim\limits_{n\to\infty}\left(\frac{1}{n+1}2^{\frac{1}{2(n+1)}}(n+1)\right)}=\frac{e}{2}\cdot\frac{e^0}{2^0}=\frac{e}{2}$$

$$Therefore\ \lim_{n\to\infty}e^{-H_n}\sum_{k=1}^{n}\frac{e^{H_k}}{\sqrt[k]{(2k-1)!!}}=\frac{e}{2}$$