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$$If\ (a_n)_{n\ge1},\quad a_1=1,\quad a_{n+1}=(n+1)!\,a_n, \forall n\in N^*\ and$$

$$(F_n)_{n\ge0},\quad F_0=0,\quad F_1=1,\quad F_{n+2}=F_{n+1}+F_n,\quad \forall n\in N$$

$$is\ Fibanocci's\ sequence, the\ compute$$

$$\lim_{n\to\infty}\frac{\sqrt[2n]{n!\cdot F_n^2}}{\sqrt[n^2]{a_n}}$$

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Solution by Amin Hajiyev-Azerbaijan

$$a_1=1, a_{n+1}=(n+1)!\,a_n\to\ a_2=2!\cdot1, a_3=2!\cdot3!\,,\dots\ a_n=\prod_{n=1}^{k}k!$$

By Binet's Formula, Fibonacci numbers exhibit asymptotic growth such that

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\phi=\frac{1+\sqrt{5}}{2}\ ,\quad F_n\sim\frac{\phi^n}{\sqrt{5}}\quad as\ n\to\infty$$

$$S=\lim_{n\to\infty}\frac{\sqrt[2n]{n!\cdot F_n^2}}{\sqrt[n^2]{a_n}}=\lim_{n\to\infty}\frac{(n!)^{\frac{1}{2n}}F_n^{\frac{1}{n}}}{a_n^{\frac{1}{n^2}}}\to\ln(S)=\lim_{n\to\infty}ln\left(\frac{(n!)^{\frac{1}{2n}}F_n^{\frac{1}{n}}}{a_n^{\frac{1}{n^2}}}\right)$$

$$\ln(S)=\lim_{n\to\infty}\left(\frac{\ln(\mathrm{n}!)}{2\mathrm{n}}+\frac{\ln(F_n)}{\mathrm{n}}-\frac{\ln(a_n)}{n^2}\right)$$

Stirling formula: $n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \ln(\mathrm{n}!)\sim\mathrm{n}\cdot\ln(n)-\mathrm{n}+\frac{1}{2}\ln(2\pi n)+O\left(\frac{1}{\mathrm{n}}\right)$

- $\ln(n!)\approx n\cdot\ln(n)-n$
- $F_n\sim\frac{\phi^n}{\sqrt{5}}\to\ \ln(F_n)\sim\mathrm{n}\ln(\phi)-\ln(\sqrt{5})$
- $a_n=\prod_{k=1}^{n}k!\to\ln(a_n)=\ln\left(\prod_{k=1}^{n}k!\right)=\sum_{k=1}^{n}\ln(k!)\approx\sum_{k=1}^{n}(k\ln(k)-k)$

Integral Approximation method: $\sum_{k=1}^{n}f(k)\sim\int_1^n f(x)dx\quad as\ n\to\infty$

$$\sum_{k=1}^{n}(k\ln(\mathrm{k})-\mathrm{k})\sim\int_1^n(x\ln(x)-x)dx=\left[\frac{x^2}{2}\ln(x)-\frac{3x^2}{4}\right]_1^n=\frac{n^2}{2}\ln(n)-\frac{3n^2}{4}$$

$$\ln(S) = \lim_{n\to\infty}\left(\frac{1}{2}\ln(n) - \frac{1}{2} + \ln(\phi) - \frac{\ln(\sqrt{5})}{\mathrm{n}} - \frac{\ln(n)}{2} + \frac{3}{4}\right) = \ln(\phi) + \frac{1}{4}$$

$$Therefore\ \ S = \lim_{n\to\infty}\frac{\sqrt[2n]{n!\cdot F_n^2}}{\sqrt[n^2]{a_n}} = e^{\ln(\phi)+\frac{1}{4}} = \phi\, e^{\frac{1}{4}} = \frac{1+\sqrt{5}}{2}\sqrt[4]{e}$$