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Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\frac{\pi}{\sqrt[n]{n!}}$$

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Solution by Amin Hajiyev-Azerbaijan

$$S=\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\frac{\pi}{\sqrt[n]{n!}}$$

$$Stirling\ formula:\ n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\sqrt[n]{n!}=(n!)^{\frac{1}{n}}=e^{\frac{1}{n}\ln(n!)}=e^{\frac{1}{n}\sum_{k=1}^{n}\ln(k)}$$

$$\sum_{k=1}^{n}\ln(k)=n\ln(\mathrm{n})-\mathrm{n}+\frac{1}{2}\ln(2\pi n)+O(1)$$

$$\sqrt[n]{n!}\approx\frac{n}{e}\left(1+\frac{\ln(2\pi n)}{2\mathrm{n}}\right)\sim\frac{n}{e}\rightarrow\ \sqrt[n+1]{(n+1)!}\sim\frac{n+1}{e}$$

$$\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\sim\frac{n+1}{e}-\frac{n}{e}=\frac{1}{e}$$

$$\sqrt[n]{(2n-1)!!}=1\bullet3\bullet5\bullet\ldots\bullet(2n-1)=\frac{(2n)!}{2^n n!}$$

$$n!\approx\left(\frac{n}{e}\right)^n,(2n)!\approx\left(\frac{2n}{e}\right)^{2n}\rightarrow(2n-1)!!\approx\frac{\left(\frac{2n}{e}\right)^{2n}}{2^n\left(\frac{n}{e}\right)^n}\approx\left(\frac{2n}{e}\right)^n$$

$$\sqrt[n]{(2n-1)!!}\approx\frac{2n}{e}$$

$$\sin\left(\frac{\pi}{\sqrt[n]{n!}}\right)=\sin(a_n)\ \lim_{n\to\infty}a_n=0\rightarrow\sin(a_n)\sim a_n$$

$$\sin\left(\frac{\pi}{\sqrt[n]{n!}}\right)\sim\frac{\pi}{\sqrt[n]{n!}}\sim\frac{e\pi}{n}$$

$$S=\lim_{n\to\infty}\left(\frac{1}{e}.\frac{2n}{e}.\frac{e\pi}{n}\right)=\lim_{n\to\infty}\frac{2e\pi}{e^2}=\frac{2\pi}{e}$$

$$Therefore\ \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\sqrt[n]{(2n-1)!!}\sin\left(\frac{\pi}{\sqrt[n]{n!}}\right)=\frac{2\pi}{e}$$