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$$\textit{Find } \lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}, \textit{where } x_n=\prod_{k=1}^{n} a_k^{\frac{1}{k}} \textit{ and } (a_n)_{n\ge1}>0$$

$$\textit{such that } \lim_{n\to\infty}\frac{a_{n+1}}{a_n\sqrt[n]{n!}}=a\in R_+^*$$

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The limit form of Stirling's Formula:

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e},\quad \lim_{n\to\infty}\frac{a_{n+1}}{na_n}=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n\sqrt[n]{n!}}\cdot\frac{\sqrt[n]{n!}}{n}\right)=\frac{a}{e}$$

Cauchy-D'Alembert criterion: $\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=y\to\lim_{n\to\infty}\sqrt[n]{b_n}=y$

$$b_n=\frac{a_n}{(n-1)!}\to\frac{b_{n+1}}{b_n}=\frac{a_{n+1}}{\frac{n!}{(n-1)!}a_n}=\frac{a_{n+1}}{na_n}, \lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\sqrt[n]{b_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{a_n}{(n-1)!}\right)}=\frac{a}{e}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{a_n}{(n-1)!}\right)}\cdot\frac{\sqrt[n]{(n-1)!}}{n}=\frac{a}{e}\cdot\frac{1}{e}=\frac{a}{e^2}$$

$$x_n=\prod_{k=1}^{n}a_k^{\frac{1}{k}}\to x_{n+1}=x_n\sqrt[n+1]{a_{n+1}}$$

$$L=\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{x_n}{n^n}}\overset{Cauchy-D'Almbert}{\cong}\lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{x_n}=$$

$$=\lim_{n\to\infty}\sqrt[n+1]{\frac{a_{n+1}}{(n+1)^{n+1}}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}\cdot\frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}=\frac{a}{e^2}\cdot\frac{1}{e}=\frac{a}{e^3}$$

$$\textit{Therefore } \lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\frac{a}{e^3}$$