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Find:

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

$$where,\lim_{n\to\infty}\frac{a_{n+1}}{na_n\sqrt[n]{(2n-1)!!}}=a\in R_+^*$$

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Solution by Amin Hajiyev-Azerbaijan

$$Stirling\ formula\quad n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$(2n-1)!!=\frac{(2n)!}{2^n n!}\sim\left(\frac{2n}{e}\right)^n\to\sqrt[n]{(2n-1)!!}\sim\frac{2n}{e}$$

Cauchy-D'Alembert criterion: $A=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\sqrt[n]{n!}$

$If\ \frac{a_{n+1}}{a_n}\sim n^k$This implies that the sequence x_n grows at an asymptotic

. rate of $(n!)^k$. $\frac{a_{n+1}}{a_n}\sim Cn^k\to a_n=a_1\cdot\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdot\ldots\cdot\frac{a_{n+1}}{a_n}$

$$a_n\sim a_1(C\cdot 1^k)(C\cdot 2^k)\ldots(C\cdot n^k)=C^n(n!)^k\sim C^n\left(\frac{n}{e}\right)^{nk}$$

$$Asyptotik\ approach\quad\sqrt[n]{a_n}\sim C\left(\frac{n}{e}\right)^k$$

$$\to\lim_{n\to\infty}\frac{a_{n+1}}{na_n\sqrt[n]{(2n-1)!!}}=a,\frac{a_{n+1}}{na_n}\sim a\sqrt[n]{(2n-1)!!},\qquad\frac{a_{n+1}}{a_n}\sim\frac{2n^2a}{e}$$

$$C=\frac{2a}{e},k=2\to\sqrt[n]{a_n}\sim\frac{2a}{e}\cdot\frac{n^2}{e^2}=\frac{2an^2}{e^3}$$

$$L=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\frac{2a(n+1)^2}{e^3}-\frac{2an^2}{e^3}\right)=$$

$$=\frac{2a}{e^3}\lim_{n\to\infty}\frac{1}{n}(n^2+2n+1-n^2)=\frac{2a}{e^3}\lim_{n\to\infty}\left(2+\frac{1}{n}\right)=\frac{4a}{e^3}$$