ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}-\frac{\sqrt[n]{n!}}{n}\right)$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$S=\lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}-\frac{\sqrt[n]{n!}}{n}\right)=\lim_{n\to\infty}nS_n(S_{n+1}-S_n)$$

$$Asymptotik\ estimate\ of\ \sqrt[n]{n!}$$

$$\sqrt[n]{n!}=e^{\frac{1}{n}\ln(n!)}=e^{\frac{1}{n}\sum_{k=1}^{n}\ln(k)}\to\sum_{k=1}^{n}\ln(k)=n\ln(n)-n+\frac{1}{2}\ln(2\pi n)+O(1)$$

$$\sqrt[n]{n!}=e^{\ln(n)-1+\frac{1}{2n}\ln(2\pi n)+O\left(\frac{1}{n}\right)}=\frac{n}{e}\left(1+\frac{\ln(2\pi n)}{2n}+O\left(\frac{1}{n}\right)\right)$$

$$S_n=\frac{\sqrt[n]{n!}}{n}\sim\frac{1}{e}$$

$$Functional\ Asymptotik\ approach:$$

To evaluate the difference $S_{n+1}-S_n$ where $S_n=\frac{\sqrt[n]{n!}}{n}$, we consider the continuous function: $f(n)=\frac{1}{e}\left(1+\frac{\ln(2\pi n)}{2n}\right)$

$$Mean\ Value\ theorem: S_{n+1}-S_n\approx\frac{d}{dn}f(n)=\frac{1-\ln(2\pi n)}{2en^2}$$

$$S=\lim_{n\to\infty}nS_n(S_{n+1}-S_n)=\lim_{n\to\infty}\frac{n}{e}\left(\frac{1-\ln(2\pi n)}{2en^2}\right)=\frac{1}{2e^2}\lim_{n\to\infty}\left(\frac{1}{n}-\frac{\ln(2\pi)}{n}-\frac{\ln(n)}{n}\right)=$$

$$=\frac{1}{2e^2}\left(\lim_{n\to\infty}\frac{1}{n}-\lim_{n\to\infty}\frac{\ln(2\pi)}{n}-\lim_{n\to\infty}\frac{\ln(n)}{n}\right)=0$$

$$Therefore\ \lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}-\frac{\sqrt[n]{n!}}{n}\right)=0$$