ROMANIAN MATHEMATICAL MAGAZINE

$$\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\int_0^{\frac{\pi}{4052}}\sqrt[\frac{n}{k}]{\frac{sen(x)+sen(3x)+sen(5x)+\cdots+sen(4051x)}{\cos(x)+\cos(3x)+\cos(5x)+\cdots+\cos(4051x)}}dx\right)\right)^{\frac{1}{2n}}=\sqrt{\frac{\pi}{a}}$$

$$a=?$$

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Solution by Yang Silva Cartolin-Peru

$$Let: S=\frac{sen(x)+sen(3x)+sen(5x)+\cdots+sen(4051x)}{cos(x)+cos(3x)+cos(5x)+\cdots+cos(4051x)}$$

$$S=\frac{\sum_{m=0}^{2025} sen\big(x+m(2x)\big)}{\sum_{m=0}^{2025} cos\big(x+m(2x)\big)}=\frac{\frac{sen^2(2026x)}{sen(x)}}{\frac{sen(2026x)cos(2026x)}{sen(x)}}=tg(2026x)$$

$$Therefore: I=\int_0^{\frac{\pi}{4052}} tg^{\frac{k}{n}}(2026x)dx$$

$$Let: u=2026x\to dx=\frac{du}{2026}\to I=\frac{1}{2026}\int_0^{\frac{\pi}{2}} tg^{\frac{k}{n}}(u)du$$

$$\#Note: \int_0^{\frac{\pi}{2}} tg^p(x)dx=\frac{\pi}{2}sec\left(\frac{p\pi}{2}\right), 0<p<1$$

$$I=\frac{\pi}{4052}sec\left(\frac{k\pi}{2n}\right)$$

$$Thus: P=\lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\frac{\pi}{4052}sec\left(\frac{k\pi}{2n}\right)\right)^{\frac{1}{2n}}\right)=\lim_{n\to\infty}\left(\left(\frac{\pi}{4052}\right)^n\prod_{k=1}^{n}sec\left(\frac{k\pi}{2n}\right)\right)^{\frac{1}{2n}}$$

$$P=\sqrt{\frac{\pi}{4052}}\lim_{n\to\infty}\left(\prod_{k=1}^{n}sec\left(\frac{k\pi}{2n}\right)\right)^{\frac{1}{2n}}$$

$$Let: L=\lim_{n\to\infty}\left(\prod_{k=1}^{n}sec\left(\frac{k\pi}{2n}\right)\right)^{\frac{1}{2n}}\to ln(L)=\lim_{n\to\infty}\frac{1}{2n}\sum_{k=1}^{n}ln\left(sec\left(\frac{k\pi}{2n}\right)\right)$$

$$ln(L)=-\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}ln\left(cos\left(\frac{k\pi}{2n}\right)\right)\to ln(L)=-\frac{1}{2}\int_0^1 ln\left(cos\left(\frac{\pi t}{2}\right)\right)dt$$

$$Let: \theta = \frac{\pi t}{2} \to dt = \frac{2}{\pi} d\theta \to ln(L) = -\frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} ln(cos(\theta))\, d\theta$$

$$ln(L) = -\frac{1}{\pi}\left(-\frac{\pi}{2} ln(2)\right) \to ln(L) = ln\left(\sqrt{2}\right) \to L = \sqrt{2}$$

$$Therefore: P = \sqrt{\frac{\pi}{4052}}\left(\sqrt{2}\right) = \sqrt{\frac{\pi}{2026}} \to a = 2026$$