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JP.599 Prove that 3 is the largest positive value of the power k such that the inequality

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\ge a_1^k+a_2^k+\cdots+a_n^k$$

holds for $n\ge 2$ and any positive real numbers $a_1,a_2,\dots,a_n$ with at most one $a_i<1$ and $a_1^2+a_2^2+\cdots+a_n^2=n$.

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Solution by proposer

For $a_1=(1+t)^{\frac{1}{2}}, a_2=\cdots=a_{n-1}=1$ and $a_n=(1-t)^{\frac{1}{2}}$, where $t\in[0,1)$, the constraints are satisfied, and the inequality is equivalent to $E(t)\ge 0$, where

$$E(t)=(1+t)^{-\frac{1}{2}}+(1-t)^{-\frac{1}{2}}+(1+t)^{\frac{k}{2}}-(1-t)^{\frac{k}{2}}.$$

We have

$$2E'(t)=-(1+t)^{-\frac{3}{2}}+(1-t)^{-\frac{3}{2}}-k(1+t)^{\frac{k}{2-1}}+k(1-t)^{\frac{k}{2-1}},$$

$$4E''(t)=3(1+t)^{-\frac{5}{2}}+3(1-t)^{-\frac{5}{2}}-k(k-2)(1+t)^{\frac{k}{2-2}}-k(k-2)(1-t)^{\frac{k}{2-2}}$$

Since $E(0)=E'(0)=0$, the condition $E''(0)\ge 0$ is necessary to have $E(t)\ge 0$ for $t\in[0,1)$.

From

$$4E''(0)=2(3-k)(1+k)\ge 0,$$

we get $k\le 3$. To show that 3 is the largest positive value of k, we need to prove that $F\ge 0$ for $n\ge 2$ and all positive a_i satisfying $a_1\le 1\le a_2\le\cdots\le a_n$ and $a_1^2+a_2^2+\cdots+a_n^2=n$,

where

$$F=\frac{1}{a_1}+\frac{1}{a_2}+\cdots.+\frac{1}{a_n}-a_1^3-a_2^3-\cdots-a_n^3.$$

We use the induction method. For $n=2$, we have

$$F=(a_1+a_2)\left(\frac{1}{a_1a_2}-a_1^2+a_1a_2-a_2^2\right)=\frac{(a_1+a_2)(a_1a_2-1)^2}{a_1a_2}\ge 0.$$

For $n \geq 3$, assume that a_1 and $a_3, \dots, a_{n-1}$ are fixed, and a_n and E are functions of a_2.

We have $a_n a_n' + a_2 = 0$ and

$$F'(a_2) = \left(-\frac{1}{a_n^2} - 3a_n^2\right) a_n' - \frac{1}{a_2^2} - 3a_2^2 = \left(\frac{1}{a_n^2} + 3a_n^2\right)\frac{a_2}{a_n} - \frac{1}{a_2^2} - 3a_2^2$$

$$= 3a_2(a_n - a_2) + \frac{a_2^3 - a_n^3}{a_2^2 a_n^3} = \frac{(a_n - a_2)\left(3a_2^3 a_n^3 - a_2^2 - a_2 a_n - a_n^2\right)}{a_2^2 a_n^3}$$

$$\geq \frac{(a_n - a_2)\left(3a_n^2 - a_2^2 - a_2 a_n - a_n^2\right)}{a_2^2 a_n^3} = \frac{(a_n - a_2)^2 (2a_n + a_2)}{a_2^2 a_n^3} \geq 0.$$

From $F'(a_2) \geq 0$, it follows that $F(a_2)$ is increasing and has the minimum value when a_2 is minimum, hence when $a_2 = 1$. So, it suffices to consider this case, when the inequality holds from the induction hypothesis. The proof is finished. If $k = 3$, then the equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.