ROMANIAN MATHEMATICAL MAGAZINE

JP.598 Let $n \geq 4$, and let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$. Prove that:

$$\frac{1}{2a_1+5} + \frac{1}{2a_2+5} + \cdots + \frac{1}{2a_n+5} \geq \frac{n}{7}$$

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Solution by proposer

Denoting

$$x = \frac{a_2 + \cdots + a_{n-1}}{n-2}, \quad S = \frac{a_1 + a_n}{2}$$

by Lemma below we have

$$(n-3)x^2 + 2Sx + a_1a_n \leq n.$$

By the AM-HM inequality,

$$\frac{1}{2a_2+5} + \cdots + \frac{1}{2a_{n-1}+5} \geq \frac{(n-2)^2}{(2a_2+5) + \cdots + (2a_{n-1}+5)} = \frac{n-2}{2x+5}$$

So, it suffices to show that

$$\frac{1}{2a_1+5} + \frac{1}{2a_n+5} + \frac{n-2}{2x+5} \geq \frac{n}{7}$$

for $a_1 \geq x \geq a_n \geq 0$ and $(n-3)x^2 + 2Sx + a_1a_n \leq n$. Since the left side of the required inequality decreases if each of a_1, a_n and x increases, it is enough to consider the case

$$(n-3)x^2 + 2Sx + a_1a_n = n.$$

Since $2S = a_1 + a_2 \geq a_1 \geq x$, we get

$$n = (n-3)x^2 + 2Sx + a_1a_n \geq (n-3)x^2 + x^2 + a_1a_n \geq (n-2)x^2,$$

hence

$$0 < x \leq x_1, x_1 = \sqrt{\frac{n}{n-2}}.$$

From $(x-a_1)(x-a_n) \leq 0$, we get $a_1a_n \leq 2Sx - x^2$, hence

$$n = (n-3)x^2 + 2Sx + a_1a_n \leq (n-3)x^2 + 2Sx + (2Sx - x^2) \leq (n-4)x^2 + 4Sx,$$

$$S \geq S_1 = \frac{n-(n-4)x^2}{4x}$$

On the other hand, since

$$\frac{1}{2a_1+5}+\frac{1}{2a_n+5}=\frac{4S+10}{4a_1a_n+20S+25}=\frac{4S+10}{4n+25-4(n-3)x^2+4(5-2x)S'}$$

the required inequality becomes

$$\frac{2S+5}{4n+25-4(n-3)x^2+4(5-2x)S}+\frac{n-7-nx}{7(2x+5)}\geq 0,$$

which can be written as

$$2A(x)S+B(x)\geq 0$$

where

$$A(x)=4nx^2-14(n-3)x+5(2n-7)=4n\left(x-\frac{7n-21}{4n}\right)^2+\frac{-9n^2+154n-441}{4n},$$

$$B(x)=4n(n-3)x^3-4(n-3)(n-7)x^2-(4n^2+25n-70)x+n(4n-3).$$

We claim that $A(x)\geq 0$. For $n\in\{4,5,\dots,13\}$, we have

$$A(x)\geq\frac{-9n^2+154n-441}{4n}>\frac{-9n^2+151n-442}{4n}=\frac{(13-n)(9n-34)}{4n}\geq 0.$$

For $n\geq 13$, since

$$x\leq x_1=\sqrt{1+\frac{2}{n-2}}<1+\frac{1}{n-2}=\frac{n-1}{n-2}<\frac{7n-21}{4n},$$

$A(x)$ is decreasing, hence

$$A(x)\geq A(x_1)=\frac{4n^2}{n-2}-14(n-3)x_1+5(2n-7)=\frac{14n^2-55n+70}{n-2}-14(n-3)x_1$$

$$>\frac{14(n-3)(n-1)}{n-2}-14(n-3)x_1=\frac{14(n-3)\left(n-1-\sqrt{n^2-2n}\right)}{n-2}>0.$$

Since $A(x)>0$, it suffices to show that $2A(x)S_1+B(x)\geq 0$. This inequality is equivalent to

$$4(n-2)x^4+6(n-3)x^3-(14n-25)x^2-6(n-6)x+5(2n-7)\geq 0,$$

$$(x-1)^2[4(n-2)x^2+2(7n-17x)+5(2n-7)]\geq 0$$

Clearly, the last inequality is true for $n\geq 4$. So, the proof is completed. The equality occurs for $a_1=a_2=\dots=a_n=1$.

Lemma. If $n\geq 4$ and $a_1,a_2,\dots,a_n$ are nonnegative real numbers such that

$a_1\geq a_2\geq\dots.\geq a_n$ and $a_1a_2+a_2a_3+\dots+a_na_1=n$, then:

$$(n-3)x^2+(a_1+a_n)x+a_1a_n\leq n,$$

where $x=\frac{a_2+\dots+a_{n-1}}{n-2}$.

Proof. Write the desired inequality in the homogeneous form:

$$(n-3)x^2+(a_1+a_n)x+a_1a_n\le a_1a_2+a_2a_3+\cdots+a_na_1,$$

which is equivalent to

$$(n-3)x^2+a_1(x-a_2)+a_n(x-a_{n-1})\le a_2a_3+\cdots+a_{n-2}a_{n-1}.$$

Since $x-a_1\le 0$ and $x-a_{n-1}\ge 0$, it suffices to show that

$$(n-3)x^2+a_2(x-a_2)+a_{n-1}(x-a_{n-1})\le a_2a_3+\cdots+a_{n-2}a_{n-1},$$

which can be rewritten as

$$a_2a_3+\cdots+a_{n-2}a_{n-1}\ge (n-3)x^2+(a_2+a_{n-1})x-a_2^2-a_{n-1}^2.$$

Since the sequences $a_2,a_3,\dots,a_{n-2}$ and $a_3,a_4,\dots,a_{n-1}$ are decreasing, by Chebyshev's inequality we have

$$(n-3)(a_2a_3+\cdots+a_{n-2}a_{n-1})\ge (a_2+\cdots+a_{n-2})(a_3+\cdots+a_{n-1})=$$
$$=\big((n-2)x-a_{n-1}\big)\big((n-2)x-a_2\big).$$

Thus, it suffices to show that

$$\frac{\big((n-2)x-a_{n-1}\big)\big((n-2)x-a_2\big)}{n-3}\ge (n-3)x^2+(a_2+a_{n-1})x-a_2^2-a_{n-1}^2,$$

which is equivalent to

$$(2n-5)x^2-(2n-5)(a_2+a_{n-1})x+(n-3)\big(a_2^2+a_{n-1}^2\big)+a_2a_{n-1}\ge 0,$$
$$(2n-5)(2x-a_2-a_{n-1})^2+(2n-7)(a_2-a_{n-1})^2\ge 0.$$

Clearly, the latter inequality is true.