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JP.597 Let $ABCD$ be an convex quadrilateral, $\lambda \in \mathbb{R}$ and M, N be such that $\overrightarrow{AN} = \lambda \cdot \overrightarrow{AB}, \overrightarrow{DN} = \lambda \cdot \overrightarrow{DC}, \overrightarrow{AD} = 3 \cdot \overrightarrow{BC}$. Find $\lambda \in \mathbb{R}$ such that $\overrightarrow{MN} = 7 \cdot \overrightarrow{BC}$

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Solution by proposers

We have:

$$\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BC} + \overrightarrow{CN} \Rightarrow \lambda \cdot \overrightarrow{MN} = \lambda\left(\overrightarrow{MB} + \overrightarrow{BC} + \overrightarrow{CN}\right) \text{ and}$$

$$(1-\lambda)\overrightarrow{MN} = (1-\lambda)\left(\overrightarrow{MA} + \overrightarrow{AD} + \overrightarrow{DN}\right), \text{ hence}$$

$$\overrightarrow{MN} = (\lambda + 1 - \lambda)\overrightarrow{MN} = \lambda\left(\overrightarrow{MB} + \overrightarrow{BC} + \overrightarrow{CN}\right) + (1-\lambda)\left(\overrightarrow{MA} + \overrightarrow{AD} + \overrightarrow{DN}\right) =$$

$$= \lambda\overrightarrow{BC} + (1-\lambda)\overrightarrow{AD} + \left[\lambda\overrightarrow{MB} + (1-\lambda)\overrightarrow{MA}\right] + \left[\lambda\overrightarrow{CN} + (1-\lambda)\overrightarrow{DN}\right]$$

Let be $CE \parallel AB, CE = AB$. From $3\overrightarrow{BC} = \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, it follows $2\overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{EC} + \overrightarrow{CD} = \overrightarrow{ED}$, then $DE \parallel BE, DE = 2BC$, and hence, $ABCD$ is an parallelogram, so $AE \parallel BC, AE = BC$.

Therefore, $\overrightarrow{MN} = \lambda\overrightarrow{BC} + (1-\lambda)\overrightarrow{AD}$ and $\overrightarrow{MN} = (3-2\lambda)\overrightarrow{BC}$, thus

$$\overrightarrow{MN} = 7\overrightarrow{BC} \Leftrightarrow 3 - 2\lambda = 7 \Leftrightarrow \lambda = -2.$$