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JP.596 In acute $\Delta ABC, AA', BB', CC'$ - are altitudes, $C' \in (AB), B' \in (AC)$, $\{H\} = BB' \cap CC'$ and E, F are middle points of $[BH], [AC]$ respectively.

Prove that:

$$4EF^2 \geq (EC' + EB')^2 + (C'F + B'F)^2$$

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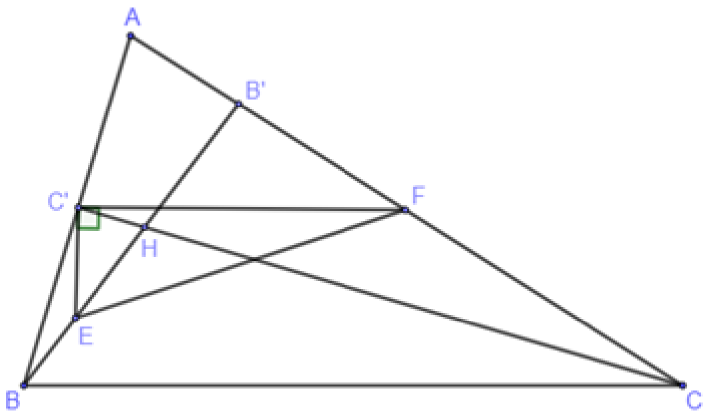
Solutions by proposers

We have:



$$\overrightarrow{C'F} \cdot \overrightarrow{C'E} = \frac{1}{2}\left(\overrightarrow{C'A} + \overrightarrow{C'C}\right) \cdot \frac{1}{2}\left(\overrightarrow{C'B} + \overrightarrow{C'H}\right) =$$

$$= \frac{1}{4}\left(\overrightarrow{C'A} \cdot \overrightarrow{C'B} + \overrightarrow{C'A} \cdot \overrightarrow{C'H} + \overrightarrow{C'C} \cdot \overrightarrow{C'B} + \overrightarrow{C'C} \cdot \overrightarrow{C'H}\right) =$$

$= \frac{1}{4}\left(\overrightarrow{C'A} \cdot \overrightarrow{C'B} + 0 + 0 + \overrightarrow{C'C} \cdot \overrightarrow{C'H}\right) = \frac{1}{4}\left(-\overrightarrow{C'A} \cdot \overrightarrow{C'B} + \overrightarrow{C'C} \cdot \overrightarrow{C'H}\right) = 0$ because $\Delta AC'C \sim$

$$\Delta HC'B \Rightarrow \frac{BC'}{C'C} = \frac{C'H}{C'A} \Leftrightarrow \overrightarrow{C'A} \cdot \overrightarrow{C'B} = \overrightarrow{C'C} \cdot \overrightarrow{C'H}$$

Hence, $\overrightarrow{C'F} \cdot \overrightarrow{C'E} = 0 \Leftrightarrow C'F \perp C'E$. So, we have:

$EF^2 = (EC')^2 + (C'F)^2$ and $EF^2 = (EB')^2 + (B'F)^2$

$$2EF = \sqrt{(EC')^2 + (C'F)^2} + \sqrt{(EB')^2 + (B'F)^2} \geq \sqrt{(EC' + EB')^2 + (C'F + B'F)^2},$$

therefore we obtain $4EF^2 \geq (EC' + EB')^2 + (C'F + B'F)^2$