ROMANIAN MATHEMATICAL MAGAZINE

JP.595 Find $x, y, z > 1$ such that:

$$\sum_{cyc} \frac{\log_2 x}{\log_2^6 x + \log_2^3 y + \log_2^3 z} = \frac{1}{27}\left(\sum_{cyc} \log_2 x\right)^3$$

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Solution by proposer

Denote: $a = \log_2 x\,; b = \log_2 y\,; c = \log_2 z$

$$x, y, z > 1 \Rightarrow a, b, c > 0$$

$$\sum_{cyc} \frac{\log_2 x}{\log_2^6 x + \log_2^3 y + \log_2^3 z} = \sum_{cyc} \frac{a}{a^6 + b^3 + c^3} \le$$

$$\overset{AM-GM}{\le} \sum_{cyc} \frac{a}{3\sqrt[3]{a^6 \cdot b^3 \cdot c^3}} = \frac{1}{3}\sum_{cyc} \frac{a}{a^2bc} = \frac{1}{3}\sum_{cyc} \frac{1}{abc} = \frac{1}{3} \cdot \frac{3}{abc} = \frac{1}{abc} \le$$

$$\overset{GM-HM}{\le} \frac{1}{\left(\frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}\right)^3} = \frac{1}{27} \cdot \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^3 =$$

$$= \frac{1}{27}\left(\log_x 2 + \log_y 2 + \log_z 2\right)^3 = \frac{1}{27}\left(\sum_{cyc} \log_x 2\right)^3$$

Equality holds for:

$$\begin{cases} a^6 = b^3 = c^3 \\ b^6 = c^3 = a^3 \\ c^6 = a^3 = b^3 \end{cases} \Rightarrow a = b = c = 1 \Rightarrow \log_2 x = \log_2 y = \log_2 z \Rightarrow x = y = z = 2$$