ROMANIAN MATHEMATICAL MAGAZINE

JP.593 Solve for real numbers:

$$\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}}=\frac{4x}{5x+2}$$

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Solution by proposer

$$(x-2)^2\geq 0\Rightarrow x^2-4x+4\geq 0\Rightarrow x^2+4\geq 4x$$

$$x+2+x^2+4\geq x+2+4x\Rightarrow 5x+2\leq x^2+x+6$$

$$\frac{1}{5x+2}\geq\frac{1}{x^2+x+6}$$

$$\frac{x+2}{5x+2}\geq\frac{x+2}{x^2+x+6}\qquad (1)$$

$$\left(\sqrt{x+2}-2\right)^2\geq 0\Rightarrow x+2-4\sqrt{x+2}+4\geq 0$$

$$x+6\geq 4\sqrt{x+2}\Rightarrow x^2+x+6\geq x^2+4\sqrt{x+2}$$

$$\frac{1}{x^2+4\sqrt{x+2}}\geq\frac{1}{x^2+x+6}$$

$$\frac{x^2}{x^2+4\sqrt{x+2}}\geq\frac{x^2}{x^2+x+6}\qquad (2)$$

$$\left(\sqrt{x+2}-x\right)^2\geq 0\Rightarrow x+2-2x\sqrt{x+2}+x^2\geq 0$$

$$x^2+x+2\geq 2x\sqrt{x+2}$$

$$x^2+x+6\geq 4+2x\sqrt{x+2}$$

$$\frac{1}{4+2x\sqrt{x+2}}\geq\frac{1}{x^2+x+6}$$

$$\frac{4}{4+2x\sqrt{x+2}}\geq\frac{4}{x^2+x+6}$$

$$\frac{2}{2+x\sqrt{x+2}}\geq\frac{4}{x^2+x+6}\qquad (3)$$

By adding (1); (2); (3):

$$\frac{x+2}{5x+2}+\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}}\geq$$

$$\geq\frac{x+2}{x^2+x+6}+\frac{x^2}{x^2+x+6}+\frac{2}{x^2+x+6}=\frac{x^2+x+6}{x^2+x+6}=1$$

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$$\frac{x+2}{5x+2}+\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}}\geq 1$$

$$\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}}\geq 1-\frac{x+2}{5x+2}$$

$$\frac{x^2}{x^2+4\sqrt{x+2}}+\frac{2}{2+x\sqrt{x+2}}\geq \frac{4x}{5x+2} \quad \textbf{(4)}$$

Equality holds in (4) for:

$$\sqrt{x+2}=x=2$$

Solution: $x=2$.