ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\int_0^1 \left(\sqrt[9]{(x^{9a}+x^{9b}+x^{9c})}\right)^5 dx \geq \frac{\sqrt[9]{4782969}}{5(a+b+c)+3}$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

$We\ know\ that\ if\ a_1, a_2, \dots, a_n\ be\ n\ positive\ real\ numbers\ and\ m\ be\ a\ rational$
$number\ then, \frac{a_1^m + a_2^m + \cdots + a_n^m}{n} > \text{ or } < \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m$
$according\ as\ m\ does\ not\ or\ does\ lie\ between\ 0\ and\ 1$
$(Reference: S.K.Mapa\ classical\ algebra\ book\ page - 22)$

$$\frac{\left((x^{9a})^{\frac{5}{9}} + (x^{9b})^{\frac{5}{9}} + (x^{9c})^{\frac{5}{9}}\right)}{3} \leq \left(\frac{x^{9a}+x^{9b}+x^{9c}}{3}\right)^{\frac{5}{9}}\ as\ \frac{5}{9} \in (0,1)$$

$$(x^{9a}+x^{9b}+x^{9c})^{\frac{5}{9}} \geq \frac{1}{3^{\frac{4}{9}}}\left((x^{9a})^{\frac{5}{9}} + (x^{9b})^{\frac{5}{9}} + (x^{9c})^{\frac{5}{9}}\right)$$

$$\left(\sqrt[9]{x^{9a}+x^{9b}+x^{9c}}\right)^5 \geq \frac{1}{3^{\frac{4}{9}}}(x^{5a}+x^{5b}+x^{5c})\ (1)$$

$$\int_0^1 \left(\sqrt[9]{(x^{9a}+x^{9b}+x^{9c})}\right)^5 dx \overset{(1)}{\geq} \frac{1}{3^{\frac{4}{9}}}\int_0^1 (x^{5a}+x^{5b}+x^{5c})\, dx =$$

$$= \frac{1}{3^{\frac{4}{9}}}\left[\left(\frac{x^{5a+1}}{5a+1}\right)_0^1 + \left(\frac{x^{5b+1}}{5b+1}\right)_0^1 + \left(\frac{x^{5c+1}}{5c+1}\right)_0^1\right] \geq \frac{1}{3^{\frac{4}{9}}}\sum \frac{1}{5a+1} \overset{Bergstrom}{\geq}$$

$$= \frac{1}{3^{\frac{4}{9}}} \cdot \frac{9}{5(a+b+c)+3} = \frac{3^{\frac{14}{5}}}{5(a+b+c)+3} =$$

$$= \frac{\sqrt[9]{3^{14}}}{5(a+b+c)+3} = \frac{\sqrt[9]{4782969}}{5(a+b+c)+3}$$

Equality holds for $a = b = c$.