

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, $ab + bc + ca = 9$ then:

$$\int_0^1 (e^{ax} + e^{bx} + e^{cx})^9 dx \leq \frac{2187(e^{9a} + e^{9b} + e^{9c} - 3)}{abc}$$

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$$e^{9ax} + e^{9bx} + e^{9cx} = \frac{(e^{ax})^9}{(1)^8} + \frac{(e^{bx})^9}{(1)^8} + \frac{(e^{cx})^9}{(1)^8} \stackrel{\text{Radon}}{\geq} \frac{(e^{ax} + e^{bx} + e^{cx})^9}{3^8} \quad (1)$$

WLOG $a \geq b \geq c$ then $bc \leq ca \leq ab$ and $e^{9ax} \geq e^{9bx} \geq e^{9cx}$

$$(bce^{9ax} + cae^{9bx} + abe^{9cx}) \stackrel{\text{Chebyshev}}{\leq} \frac{1}{3}(e^{9ax} + e^{9bx} + e^{9cx})(ab + bc + ca) =$$

$$= \frac{1}{3} \times (e^{9ax} + e^{9bx} + e^{9cx}) = 3(e^{9ax} + e^{9bx} + e^{9cx}) \quad (2)$$

$$\begin{aligned} \int_0^1 (e^{ax} + e^{bx} + e^{cx})^9 dx &\stackrel{(1)}{\leq} 3^8 \int_0^1 (e^{9ax} + e^{9bx} + e^{9cx}) dx = \\ &= 3^8 \left(\frac{e^{9ax}}{9a} + \frac{e^{9bx}}{9b} + \frac{e^{9cx}}{9c} \right)_0^1 = 3^6 \left(\frac{e^{9a} - 1}{a} + \frac{e^{9b} - 1}{b} + \frac{e^{9c} - 1}{c} \right) = \\ &= 3^6 \left(\frac{3(e^{9ax} + e^{9bx} + e^{9cx}) - (ab + bc + ca)}{abc} \right) \leq \end{aligned}$$

$$\stackrel{(2)}{\leq} 3^6 \left(\frac{3(e^{9ax} + e^{9bx} + e^{9cx}) - 9}{abc} \right) = 3^7 \left(\frac{(e^{9ax} + e^{9bx} + e^{9cx}) - 3}{abc} \right) =$$

$$= 2187 \left(\frac{(e^{9ax} + e^{9bx} + e^{9cx}) - 3}{abc} \right)$$

Equality holds for $a = b = c = \sqrt{3}$.