

ROMANIAN MATHEMATICAL MAGAZINE

Let $0 < a \leq b$ and let $f: [a^k, b^k] \rightarrow \mathbb{R}$ be a continuous function where $k \geq 2$ is a positive integer. Prove that:

$$\frac{b^{2k-1+2m} - a^{2k-1+2m}}{2k-1+2m} + \int_a^b x^{2m} f^2(x^k) dx \geq \frac{2}{k} \int_{a^k}^{b^k} y^{\frac{2m}{k}} f(y) dy$$

for any given integer parameter $m \geq 0$.

Proposed by Omkumar Rao-India

Solution by proposer

$$\frac{b^{2k-1+2m} - a^{2k-1+2m}}{\int_a^b x^{2k-2+2m} dx} + \int_a^b x^{2m} f^2(x^k) dx \geq \frac{2}{k} \int_{a^k}^{b^k} y^{\frac{2m}{k}} f(y) dy$$

$$\int_a^b (x^{2m} f^2(x^k) + x^{2k-2+2m}) dx \geq \frac{2}{k} \underbrace{\int_{a^k}^{b^k} y^{\frac{2m}{k}} f(y) dy}_{y=x^k}$$

$$\int_a^b (x^{2m} f^2(x^k) + x^{2k-2+2m}) dx \geq \int_a^b 2x^{k-1+2m} f(x^k) dx$$

$$\int_a^b x^{2m} (x^{k-1} - f(x^k))^2 dx \geq 0 \text{ as } x^{2m} \geq 0 \text{ and } (x^{k-1} - f(x^k))^2 \geq 0 \text{ for all } x \in [a, b]$$