

ROMANIAN MATHEMATICAL MAGAZINE

Let $m, n \in \mathbb{N}$ with $n > m \geq 1$ and let $g: [0, 1] \rightarrow [0, \infty)$ be a non-negative continuous function on $[0, 1]$ such that $x^m(1 - x^n)g(x) \leq x^2$ for all $x \in [0, 1]$. Prove that:

$$1 \leq \int_0^1 \frac{dx}{\sqrt{1 - x^2 + x^m(1 - x^n)g(x)}} \leq \frac{\pi}{2}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\begin{aligned} x^m(1 - x^n)g(x) \leq x^2 &\Rightarrow 1 - x^2 + x^m(1 - x^n)g(x) \leq 1 \\ \frac{1}{\sqrt{1 - x^2 + x^m(1 - x^n)g(x)}} &\geq 1 \Rightarrow \int_0^1 \frac{dx}{\sqrt{1 - x^2 + x^m(1 - x^n)g(x)}} \geq 1 \\ x^m(1 - x^n)g(x) \geq 0 &\Rightarrow 1 - x^2 + x^m(1 - x^n)g(x) \geq 1 - x^2 \\ \frac{1}{\sqrt{1 - x^2 + x^m(1 - x^n)g(x)}} &\leq \frac{1}{\sqrt{1 - x^2}} \Rightarrow \int_0^1 \frac{dx}{\sqrt{1 - x^2 + x^m(1 - x^n)g(x)}} \leq \\ &\leq \int_0^1 \frac{1}{\sqrt{1 - x^2}} dx = \frac{\pi}{2} \end{aligned}$$

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