ROMANIAN MATHEMATICAL MAGAZINE

$$If\ 0 < a \le b\ then\ show\ that:$$

$$\int_a^b \int_a^b \frac{dxdy}{\left(x\sqrt{x}+y\sqrt{y}\right)^2} \le \frac{b-a}{8}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x\sqrt{x}+y\sqrt{y} \overset{AM-GM}{\ge} 2\sqrt{x^{\frac{3}{2}}y^{\frac{3}{2}}} = 2x^{\frac{3}{4}}y^{\frac{3}{4}}\ (1)$$

$$\int_a^b \int_a^b \frac{dxdy}{\left(x\sqrt{x}+y\sqrt{y}\right)^2} \overset{(1)}{\le} \int_a^b \int_a^b \frac{dxdy}{4x^{\frac{3}{2}}y^{\frac{3}{2}}} = \frac{1}{4}\int_a^b \int_a^b x^{-\frac{3}{2}}y^{-\frac{3}{2}}dxdy =$$

$$= \frac{1}{4}\int_a^b x^{-\frac{3}{2}}dx \times \int_a^b y^{-\frac{3}{2}}dy = \frac{1}{4}\left(\left(-\frac{2}{\sqrt{x}}\right)_a^b\left(-\frac{2}{\sqrt{y}}\right)_a^b\right) = \left(\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{b}}\right)^2$$

$$We\ need\ to\ show:$$

$$\left(\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{b}}\right)^2 \le \frac{b-a}{8}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$8ab\left(\sqrt{b}-\sqrt{a}\right)^2 \le (b-a)(b+a)(b-a)\ or, 8ab\left(\sqrt{b}-\sqrt{a}\right)^2 \le (b-a)^2(b+a)$$

$$Let\ b = x^2\ \&\ a = y^2 > 0$$

$$L.H.S = 8ab\left(\sqrt{b}-\sqrt{a}\right)^2 = 8x^2y^2(x-y)^2$$

$$R.H.S = (b-a)^2(b+a) = (x^2-y^2)^2(x^2+y^2) = (x+y)^2(x-y)^2(x^2+y^2) \overset{AM-GM}{\ge}$$

$$\ge (4xy)(x-y)^2 2xy = 8xy(x-y)^2\ so\ L.H.S \le R.H.S$$

Equality holds for a=b.