

ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dx dy}{\sqrt{x+y}} \leq \sqrt[4]{8}(b-a)(\sqrt{b}-\sqrt{a})$$

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$$\begin{aligned} x+y &\stackrel{AM-GM}{\geq} 2\sqrt{xy} \text{ \& } \sqrt{x}+\sqrt{y} \stackrel{AM-GM}{\geq} 2\sqrt[4]{xy} \\ (\sqrt{x}+\sqrt{y})^2 &\geq 4\sqrt{xy} \\ (x+y)(\sqrt{x}+\sqrt{y})^2 &\geq 2\sqrt{xy} \cdot 4\sqrt{xy} = 8xy \\ \sqrt{x+y}(\sqrt{x}+\sqrt{y}) &\geq \sqrt{8}\sqrt{xy} \\ \frac{(\sqrt{x}+\sqrt{y})}{\sqrt{xy}} &\geq \sqrt{8} \cdot \frac{1}{\sqrt{x+y}} > 2^{\frac{5}{4}} \cdot \frac{1}{\sqrt{x+y}} \left(\text{as } 2^{\frac{5}{4}} < \sqrt{8} \right) \\ \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} &\geq 2^{\frac{5}{4}} \cdot \frac{1}{\sqrt{x+y}} \\ \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} \right) \left(\frac{1}{2^{\frac{5}{4}}} \right) &\geq \frac{1}{\sqrt{x+y}} \quad (A) \\ \int_a^b \int_a^b \frac{dx dy}{\sqrt{x+y}} &\stackrel{\text{by (A)}}{\leq} \frac{1}{2^{\frac{5}{4}}} \int_a^b \int_a^b \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} \right) dx dy = \\ &= \frac{1}{2^{\frac{5}{4}}} \left(\int_a^b \int_a^b \left(\frac{1}{\sqrt{x}} \right) dx dy + \int_a^b \int_a^b \left(\frac{1}{\sqrt{y}} \right) dx dy \right) = \\ &= \frac{1}{2^{\frac{5}{4}}} \left(2(\sqrt{x})_a^b (y)_a^b + 2(\sqrt{y})_a^b (x)_a^b \right) = \frac{4}{2^{\frac{5}{4}}} (b-a)(\sqrt{b}-\sqrt{a}) = \\ &= \frac{3}{2^{\frac{5}{4}}} (b-a)(\sqrt{b}-\sqrt{a}) = \sqrt[4]{8}(b-a)(\sqrt{b}-\sqrt{a}) \end{aligned}$$

Equality holds for $a=b$.