

ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \leq b$ then:

$$\int_a^b \int_a^b \frac{dx dy}{(\sqrt[3]{x} + \sqrt[3]{y})^2} \leq \frac{3}{4} (b - a) (\sqrt[3]{b} - \sqrt[3]{a})$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x, y > 0 \text{ then } (\sqrt[3]{x} + \sqrt[3]{y})^2 \stackrel{AM-GM}{\geq} 4\sqrt[3]{xy} \text{ or, } \frac{1}{(\sqrt[3]{x} + \sqrt[3]{y})^2} \leq \frac{1}{4\sqrt[3]{xy}}$$

$$\begin{aligned} \text{now, } \int_a^b \int_a^b \frac{dx dy}{(\sqrt[3]{x} + \sqrt[3]{y})^2} &\leq \frac{1}{4} \int_a^b \int_a^b \frac{dx dy}{\sqrt[3]{xy}} = \frac{1}{4} \int_a^b \frac{dx}{\sqrt[3]{x}} \cdot \int_a^b \frac{dx}{\sqrt[3]{y}} = \\ &= \frac{1}{4} \left(\frac{3}{2}\right)^2 \left(b^{\frac{2}{3}} - a^{\frac{2}{3}}\right)^2 = \frac{9}{16} \left(b^{\frac{2}{3}} - a^{\frac{2}{3}}\right)^2 \end{aligned}$$

We need to show:

$$\frac{9}{16} \left(b^{\frac{2}{3}} - a^{\frac{2}{3}}\right)^2 \leq \frac{3}{4} (b - a) (\sqrt[3]{b} - \sqrt[3]{a})$$

$$\text{or, } 3 \left(b^{\frac{2}{3}} - a^{\frac{2}{3}}\right)^2 \leq 4(b - a) (\sqrt[3]{b} - \sqrt[3]{a}) \text{ or, } 3(u^2 - v^2)^2 \stackrel{b=u^3, b=v^3}{\leq} \stackrel{b>u \text{ or } u^3>v^3}{\leq} 4(u^3 - v^3)(u - v)$$

$$\text{or, } 3(u + v)^2 (u - v)^2 \leq 4(u - v)^2 (u^2 + uv + v^2) \text{ or, } 3(u + v)^2 \leq 4(u^2 + uv + v^2)$$

$$\text{true as } 4(u^2 + uv + v^2) = 3(u^2 + v^2) + (u^2 + v^2) + 4uv \stackrel{AM \geq GM}{\geq}$$

$$\geq 3(u^2 + v^2) + 2uv + 4uv = 3(u + v)^2$$

Equality holds for $a = b$.