ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^{\frac{\pi}{2}} \frac{2(2sinx+3cosx)}{2cosx+3sinx}dx > \pi$$

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$Lemma: \forall\, x>0, \ln(1+x) > \frac{x}{1+x}$

$Proof: let\ f(x) = \ln(1+x) - \frac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0\ so\ f(x)\ is\ increasing, f(0) = 0$$

$$and\ f(x) > f(0)\ or, \ln(1+x) - \frac{x}{1+x} > 0\ or, \ln(1+x) > \frac{x}{1+x}\ \ (1)$$

$$\ln\frac{3}{2} = \ln\left(1+\frac{1}{2}\right) \overset{(1)}{>} \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}\ (2)$$

$$clearly\ 10 > \frac{66}{7} = 3\times\frac{22}{7} = 3\pi\ (3)$$

$$let\ 4sinx + 6cosx = m(2cosx+3sinx) + n.\frac{d}{dx}(2cosx+3sinx)$$

$$4sinx + 6cosx = m(2cosx+3sinx) + n(3cosx-2sinx)$$

$$4sinx + 6cosx = (3m-2n)sinx + (2m+3n)cosx$$

$$comparing\ the\ coeficient\ of\ sinx\ \&\ cosx\ we\ get$$

$$2m+3n = 6, 3m-2n = 4\ , solving\ we\ get\ m = \frac{24}{13}, n = \frac{10}{13}$$

$$\int_0^{\frac{\pi}{2}} \frac{2(2sinx+3cosx)}{2cosx+3sinx}dx = \int_0^{\frac{\pi}{2}} \frac{(4sinx+6cosx)}{2cosx+3sinx}dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{24}{13}(2cosx+3sinx) + \frac{10}{13}(3cosx-2sinx)}{2cosx+3sinx}dx =$$

$$= \left(\frac{24}{13}x + \frac{10}{13}\ln|2cosx+3sinx\,|\right)_0^{\frac{\pi}{2}} = \frac{12\pi}{13} + \frac{10}{13}\ln\frac{3}{2} \overset{(2)\&(3)}{>} \frac{12\pi}{13} + \frac{3\pi}{13}.\frac{1}{3} = \frac{13\pi}{13} = \pi$$