ROMANIAN MATHEMATICAL MAGAZINE

$$If\ 0 < a \le b\ then:$$

$$\int_a^b \int_a^b \int_a^b \frac{z(x+y)^2 + x(y+z)^2 + y(x+z)^2}{(x+y+z)^3} dxdydz \le \frac{4}{9}(b-a)^3$$

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$$A = z(x+y)^2 + x(y+z)^2 + y(x+z)^2$$

$$= z(x^2+y^2+2xy) + x(y^2+z^2+2yz) + y(x^2+z^2+2xz) = \sum_{sym} x^2y + 6xyz$$

$$B = (x+y+z)^3 = x^3+y^3+z^3+3(x+y)(y+z)(z+x) = \sum x^3 + 3\sum_{sym} x^2y + 6xyz$$

$$We\ will\ show \frac{A}{B} \le \frac{4}{9}, 9A \le 4B, 9\left(\sum_{sym} x^2y + 6xyz\right) \le 4\left(\sum x^3 + 3\sum_{sym} x^2y + 6xyz\right)$$

$$or, 4\sum x^3 + 3\sum_{sym} x^2y - 30xyz \ge 0\ True$$

$$since, 4\sum x^3 \overset{AM-GM}{\ge} 12xyz, 3\sum_{sym} x^2y \overset{AM-GM}{\ge} 3 \times 6xyz = 18xyz\ then$$

$$4\sum x^3 + 3\sum_{sym} x^2y - 30xyz \ge 12xyz + 18xyz - 30xyz \ge 0$$

$$so \frac{A}{B} = \frac{z(x+y)^2 + x(y+z)^2 + y(x+z)^2}{(x+y+z)^3} \le \frac{4}{9}$$

$$now, \int_a^b \int_a^b \int_a^b \frac{z(x+y)^2 + x(y+z)^2 + y(x+z)^2}{(x+y+z)^3} dxdydz \le \int_a^b \int_a^b \int_a^b \frac{4}{9} dxdydz =$$

$$= \frac{4}{9}(b-a)^3$$

Equality holds for a=b