

ROMANIAN MATHEMATICAL MAGAZINE

For any $x, y, z > 0$, prove that:

$$\prod_{\text{cyc}} \frac{\Gamma(x+y)}{\Gamma(x)} \leq \frac{2^{2(x+y+z)}}{\pi\sqrt{\pi}} \prod_{\text{cyc}} \int_0^\infty t^{2x} e^{-t^2} dt$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Nawar Alasadi-Iraq

$$\begin{aligned} \int_0^\infty t^{2x} e^{-t^2} dt &= \frac{1}{2} \int_0^\infty u^{x-\frac{1}{2}} e^{-u} du = \frac{1}{2} \Gamma\left(x + \frac{1}{2}\right) \\ \text{RHS} &= \frac{2^{2(x+y+z)}}{\pi\sqrt{\pi}} \prod_{\text{cyc}} \frac{1}{2} \Gamma\left(x + \frac{1}{2}\right) \\ &= \frac{2^{2(x+y+z)}}{8\pi^{3/2}} \Gamma\left(x + \frac{1}{2}\right) \Gamma\left(y + \frac{1}{2}\right) \Gamma\left(z + \frac{1}{2}\right) \\ \Gamma(x) \Gamma\left(x + \frac{1}{2}\right) &= 2^{1-2x} \sqrt{\pi} \Gamma(2x) \\ \Gamma\left(x + \frac{1}{2}\right) &= \frac{2^{1-2x} \sqrt{\pi} \Gamma(2x)}{\Gamma(x)} \\ \text{RHS} &= \frac{2^{2(x+y+z)}}{8\pi^{3/2}} \prod_{\text{cyc}} \frac{2^{1-2x} \sqrt{\pi} \Gamma(2x)}{\Gamma(x)} \\ &= \frac{2^{2(x+y+z)}}{2^3 \pi^{3/2}} (2^{3-2(x+y+z)} \pi^{3/2}) \frac{\Gamma(2x) \Gamma(2y) \Gamma(2z)}{\Gamma(x) \Gamma(y) \Gamma(z)} \\ &= \frac{\Gamma(2x) \Gamma(2y) \Gamma(2z)}{\Gamma(x) \Gamma(y) \Gamma(z)} \\ \text{LHS} \leq \text{RHS} &\Leftrightarrow \prod_{\text{cyc}} \frac{\Gamma(x+y)}{\Gamma(x)} \leq \frac{\Gamma(2x) \Gamma(2y) \Gamma(2z)}{\Gamma(x) \Gamma(y) \Gamma(z)} \\ &\Leftrightarrow \Gamma(x+y) \Gamma(y+z) \Gamma(z+x) \leq \Gamma(2x) \Gamma(2y) \Gamma(2z) \\ f(t) &= \ln \Gamma(t) \Rightarrow f''(t) = \psi'(t) > 0 \Rightarrow f \text{ is strictly convex on } (0, \infty) \\ \text{W.L.O.G. } x &\geq y \geq z > 0 \\ A &= (2x, 2y, 2z) \quad \text{and} \quad B = (x+y, x+z, y+z) \\ 2x &\geq x+y \\ 2x+2y &\geq (x+y) + (x+z) \\ 2x+2y+2z &= (x+y) + (x+z) + (y+z) \\ \Rightarrow A &> B \\ \text{Karamata's Inequality} &\Rightarrow f(2x) + f(2y) + f(2z) \geq f(x+y) + f(x+z) + f(y+z) \\ &\Rightarrow \ln(\Gamma(2x) \Gamma(2y) \Gamma(2z)) \geq \ln(\Gamma(x+y) \Gamma(x+z) \Gamma(y+z)) \\ &\Rightarrow \Gamma(2x) \Gamma(2y) \Gamma(2z) \geq \Gamma(x+y) \Gamma(y+z) \Gamma(z+x) \end{aligned}$$