ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_{-\infty}^{\infty} \frac{\operatorname{sech}(x)}{\sec(x)}\,dx$$

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$$I = \int_{-\infty}^{\infty} \frac{\operatorname{sech}(x)}{\sec(x)}\,dx = 2\int_{0}^{\infty} \frac{\cos(x)}{\cosh(x)}\,dx$$

$$I = 4\int_{0}^{\infty} \frac{\cos(x)}{1+e^{-2x}}e^{-x}\,dx$$

$$= 4\sum_{n=0}^{\infty}(-1)^n \int_{0}^{\infty} \cos(x)\, e^{-(2n+1)x}dx \quad \text{(Dominated Convergence Theorem)}$$

Justification: For x > 0 let

$$S_N(x) = \sum_{n=0}^{N-1}(-1)^n e^{-(2n+1)x} = \frac{e^{-x}(1-(-1)^N e^{-2Nx})}{1+e^{-2x}} \to \frac{e^{-x}}{1+e^{-2x}} \quad \text{as N}\to\infty$$

$$|\cos(x)\, S_N(x)| \le 2e^{-x}$$

which is integrable on (0, ∞)

$$I = 4\sum_{n=0}^{\infty}(-1)^n \frac{2n+1}{(2n+1)^2+1} = 4\Re\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1+i}$$

$$I = 2\Re\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1+i}{2}} = \Re\left\{\psi\left(\frac{3+i}{4}\right)-\psi\left(\frac{1+i}{4}\right)\right\}$$

$$\text{But } \overline{\psi(z)} = \psi(\bar{z}) \;\therefore\; \Re\psi(z) = \Re\psi(\bar{z})$$

$$\therefore\; \Re\psi\left(\frac{3+i}{4}\right) = \Re\psi\left(\frac{3-i}{4}\right) = \Re\psi\left(1-\frac{1+i}{4}\right)$$

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$$\psi(1-z)-\psi(z)=\pi\cot(\pi z)$$

$$\therefore\ I=\Re\left\{\psi\left(1-\frac{1+i}{4}\right)-\psi\left(\frac{1+i}{4}\right)\right\}$$

$$\therefore\ I=\Re\left\{\pi\cot\left(\frac{\pi}{4}(1+i)\right)\right\}=\frac{\pi\sin\left(2\frac{\pi}{4}\right)}{\cosh\left(2\frac{\pi}{4}\right)-\cos\left(2\frac{\pi}{4}\right)}$$

$$I=\frac{\pi}{\cosh\left(\frac{\pi}{2}\right)}$$