

Find:

$$I = \int_0^1 \frac{\ln(x) \ln(-\ln(x))}{1+x^2} dx$$

Proposed by Zinadin Kebaili-Algeria

Solution by Samuel Ajibade-Nigeria

$$\begin{aligned}
 I &= \int_0^1 \frac{\ln(x) \ln(-\ln(x))}{1+x^2} dx \\
 t &= -\ln(x), x = e^{-t}, dx = -e^{-t} dt, t \in [0, \infty) \\
 I &= \int_0^\infty \frac{-t \ln(t)}{1+e^{-2t}} e^{-t} dt = \int_0^\infty t \ln(t) e^{-t} \sum_{n=0}^\infty (-1)^{n-1} e^{-2nt} dt \\
 I &= \sum_{n=0}^\infty (-1)^{n-1} \int_0^\infty t \ln(t) e^{-(2n+1)t} dt \\
 I &= \sum_{n=0}^\infty (-1)^{n-1} \frac{\partial}{\partial a} \int_0^\infty t \cdot t^{a-1} e^{-(2n+1)t} dt \Big|_{a=1} \\
 I &= \sum_{n=0}^\infty (-1)^{n-1} \frac{\partial}{\partial a} \int_0^\infty t^a e^{-(2n+1)t} dt \Big|_{a=1} \\
 I &= \sum_{n=0}^\infty (-1)^{n-1} \frac{\partial}{\partial a} \frac{\Gamma(a+1)}{(2n+1)^{a+1}} \Big|_{a=1} \\
 I &= \sum_{n=0}^\infty (-1)^{n-1} \frac{\Gamma(a+1)}{(2n+1)^{a+1}} (\psi(a+1) - \ln(2n+1)) \Big|_{a=1} \\
 I &= \sum_{n=0}^\infty (-1)^{n-1} \frac{\Gamma(2)}{(2n+1)^2} (\psi(2) - \ln(2n+1)) \\
 I &= \sum_{n=0}^\infty (-1)^n \frac{\ln(2n+1)}{(2n+1)^2} - \sum_{n=0}^\infty (-1)^n \frac{\psi(2)}{(2n+1)^2} \\
 I &= \frac{\partial}{\partial b} \sum_{n=0}^\infty (-1)^n \frac{1}{(2n+1)^{2-b}} \Big|_{b=0} - \psi(2) \sum_{n=0}^\infty (-1)^n \frac{1}{(2n+1)^2} \\
 I &= \frac{\partial}{\partial b} \beta(2-b) \Big|_{b=0} - \psi(2) \beta(2) \\
 &\quad - \beta'(2) - (1-\gamma)G
 \end{aligned}$$

$$I = (\gamma - 1)G - \beta'(2)$$

$\gamma = \text{Euler - Mascheroni constant}, G = \beta(2) = \text{Catalan's Constant}$