

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_0^{\infty} e^{-\frac{t}{2}} \ln^3(t) \sin\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}t\right) dt$$

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$$\begin{aligned} K &= \int_0^{\infty} e^{-\frac{t}{2}} \ln^3(t) e^{i\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}t\right)} dt \\ &: I = \Im(K) \quad \dots (1) \\ K &= e^{i\frac{\pi}{3}} \int_0^{\infty} e^{-\frac{t}{2}} \ln^3(t) e^{-i\frac{\sqrt{3}}{2}t} dt = e^{i\frac{\pi}{3}} \int_0^{\infty} e^{-\frac{t}{2}(1+i\sqrt{3})} \ln^3(t) dt \\ &\text{Let } z = \frac{1+i\sqrt{3}}{2} \\ K &= e^{i\frac{\pi}{3}} \int_0^{\infty} e^{-zt} \ln^3(t) dt = e^{i\frac{\pi}{3}} \frac{\partial^3}{\partial a^3} \int_0^{\infty} t^a e^{-zt} dt \Big|_{a=0} \\ K &= \frac{e^{i\frac{\pi}{3}}}{z} \frac{\partial^3}{\partial a^3} \frac{\Gamma(a+1)}{z^a} \Big|_{a=0} = \frac{e^{i\frac{\pi}{3}}}{z} \frac{\partial^3}{\partial a^3} \Gamma(a+1) z^{-a} \Big|_{a=0} \end{aligned}$$

By Leibnitz's theorem for nth derivatives: $(u \cdot v)^{(n)} = \sum_{r=0}^n \binom{n}{r} u^{(n-r)} v^{(r)}$

$$\begin{aligned} \frac{\partial^3}{\partial a^3} \Gamma(a+1) z^{-a} &= \sum_{r=0}^3 \binom{3}{r} (\Gamma(a+1))^{(n-r)} (z^{-a})^{(r)} \\ \frac{\partial^3}{\partial a^3} \Gamma(a+1) z^{-a} \Big|_{a=0} &= \Gamma^{(3)}(a+1) z^{-a} + 3\Gamma^{(2)}(a+1)(z^{-a})^{(1)} + 3\Gamma^{(1)}(a+1)(z^{-a})^{(2)} + \Gamma(a+1)(z^{-a})^{(3)} \Big|_{a=0} \\ : \Gamma^{(1)}(a+1) &= \Gamma(a+1)\psi(a+1), \Gamma^{(2)}(a+1) = \Gamma(a+1)(\psi^2(a+1) + \psi^{(1)}(a+1)) \\ \Gamma^{(3)}(a+1) &= \Gamma(a+1)\psi(a+1)(\psi^2(a+1) + \psi^{(1)}(a+1)) + \Gamma(a+1)(2\psi(a+1)\psi^{(1)}(a+1) + \psi^{(2)}(a+1)) \\ : (z^{-a})^{(1)} &= -z^{-a}\ln(z), (z^{-a})^{(2)} = z^{-a}\ln^2(z), (z^{-a})^{(3)} = -z^{-a}\ln^3(z) \\ &\vdots \\ \frac{\partial^3}{\partial a^3} \Gamma(a+1) z^{-a} \Big|_{a=0} &= \Gamma(1)[\psi(1)(\psi^2(1) + 3\psi^{(1)}(1)) + \psi^{(2)}(1)] - 3\ln(z)\Gamma(1)(\psi^2(1) + \psi^{(1)}(1)) + 3\ln^2(z)\Gamma(1)\psi(1) - \ln^3(z)\Gamma(1) \end{aligned}$$

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$$\therefore \Gamma(1) = 1, \psi(1) = -\gamma, \psi^{(1)}(1) = \frac{\pi^2}{6}, \psi^{(2)}(1) = -2\zeta(3)$$

$$\left. \frac{\partial^3}{\partial a^3} \Gamma(a+1)z^{-a} \right|_{a=0}$$

$$= -\gamma \left(\gamma^2 + \frac{\pi^2}{2} \right) - 2\zeta(3) - 3\ln(z) \left(\gamma^2 + \frac{\pi^2}{6} \right) - 3\gamma \ln^2(z) - \ln^3(z)$$

$$e^{i\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1 + i\sqrt{3}}{2} = z$$

$$\therefore \ln(z) = \ln(e^{i\frac{\pi}{3}}) = i\frac{\pi}{3}$$

$\therefore$

$$K = -\frac{z}{z} \left[\gamma \left(\gamma^2 + \frac{\pi^2}{2} \right) + 2\zeta(3) + 3 \cdot i \frac{\pi}{3} \left(\gamma^2 + \frac{\pi^2}{6} \right) + 3\gamma \left(i \frac{\pi}{3} \right)^2 + \left(i \frac{\pi}{3} \right)^3 \right]$$

$$K = - \left[\gamma \left(\gamma^2 + \frac{\pi^2}{2} \right) + 2\zeta(3) + i\pi \left(\gamma^2 + \frac{\pi^2}{6} \right) - \gamma \frac{\pi^2}{3} + -i \frac{\pi^3}{27} \right]$$

From equation 1: $I = \Im(K)$

$$I = -\pi \left(\gamma^2 + \frac{\pi^2}{6} \right) + \frac{\pi^3}{27} = -\pi \left(\gamma^2 + \frac{\pi^2}{6} - \frac{\pi^2}{27} \right) = -\frac{\pi}{54} (54\gamma^2 + 7\pi^2)$$

$$I = -\frac{\pi}{54} (54\gamma^2 + 7\pi^2)$$

Where: ' γ ' is the Euler – Mascheroni constant