

Find:

$$I = \int_0^{\frac{\pi}{2}} \ln(a^2 + b^2 \tan^2(x)) dx$$

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$$I = \int_0^{\frac{\pi}{2}} \ln(a^2 + b^2 \tan^2(x)) dx = \int_0^{\frac{\pi}{2}} \ln\left(a^2 \left(1 + \left(\frac{b}{a}\right)^2 \tan^2(x)\right)\right) dx$$

$$I = \int_0^{\frac{\pi}{2}} \ln(a^2) dx + \int_0^{\frac{\pi}{2}} \ln\left(1 + \left(\frac{b}{a}\right)^2 \tan^2(x)\right) dx$$

$$I = \pi \ln|a| + J \quad \dots \text{equation 1}$$

$$J = \int_0^{\frac{\pi}{2}} \ln(1 + c \tan^2(x)) dx \quad \dots \left\{ \left(\frac{b}{a}\right)^2 = c \right\}$$

$$J = \int_0^c \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial c} \ln(1 + c \tan^2(x)) dx dc = \int_0^c \int_0^{\frac{\pi}{2}} \frac{\tan^2(x)}{1 + c \tan^2(x)} dx dc$$

$$J = \int_0^c \frac{1}{c} \int_0^{\frac{\pi}{2}} \frac{1 + c \tan^2(x) - 1}{1 + c \tan^2(x)} dx dc$$

$$J = \int_0^c \frac{1}{c} \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{1 + c \tan^2(x)}\right) dx dc \quad \dots (2)$$

Solving the inner integral, say K

$$K = \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{1 + c \tan^2(x)}\right) dx = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{1}{1 + c \tan^2(x)} dx$$

$$\text{let } t = \tan(x), dx = \frac{dt}{1 + t^2}, t \in [0, \infty)$$

$$K = \frac{\pi}{2} - \int_0^{\infty} \frac{1}{(1 + ct^2)(1 + t^2)} dx$$

$$K = \frac{\pi}{2} + \frac{1}{1 - c} \int_0^{\infty} \left(\frac{c}{1 + ct^2} - \frac{1}{1 + t^2}\right) dx$$

$$K = \frac{\pi}{2} + \frac{1}{1-c} \left(\sqrt{c} (\tan^{-1}(ct) - \tan^{-1}(t)) \right) \Big|_0^{\infty}$$

$$K = \frac{\pi}{2} + \frac{\pi \sqrt{c} - 1}{2(1-c)} = \frac{\pi}{2} \left(\frac{\sqrt{c} - c}{1-c} \right)$$

Solving for J from equation 2

$$J = \frac{\pi}{2} \int_0^c \frac{\sqrt{c} - c}{c(1-c)} dc = \frac{\pi}{2} \int_0^c \frac{1 - \sqrt{c}}{\sqrt{c}(1 - \sqrt{c})(1 + \sqrt{c})} dc$$

$$J = \frac{\pi}{2} \int_0^c \frac{1}{\sqrt{c}(1 + \sqrt{c})} dc$$

$$\text{let } u = 1 + \sqrt{c}, du = \frac{1}{2\sqrt{c}} dc, u \in [1, 1 + \sqrt{c}]$$

$$J = \pi \int_1^{1+\sqrt{c}} \frac{1}{u} du = \pi \ln(1 + \sqrt{c}) \cdots \left\{ \left(\frac{b}{a} \right)^2 = c \right\}$$

$$J = \pi \ln \left(1 + \frac{|b|}{|a|} \right) = \pi \ln(|a| + |b|) - \pi \ln|a|$$

Expressing from I equation 1

$$I = \pi \ln(|a| + |b|) \\ \forall a, b \in \mathbb{R}$$