

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_0^1 \frac{\ln^2(x)}{(x-1)x^a} dx$$

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$$I = \int_0^1 \frac{\ln^2(x)}{(x-1)x^a} dx = -\frac{\partial^2}{\partial b^2} \int_0^1 \frac{x^{b-a}}{1-x} dx \Big|_{b=0} \quad (\text{Leibniz's rule})$$

$$I = -\frac{\partial^2}{\partial b^2} \int_0^1 \sum_{n=0}^{\infty} x^{n+b-a} dx \Big|_{b=0} \quad (\text{Geometric series: } |x| < 1)$$

$$I = -\sum_{n=0}^{\infty} \frac{\partial^2}{\partial b^2} \int_0^1 x^{n+b-a} dx \Big|_{b=0} \quad (\text{Fubini - Tonelli Theorem})$$

$$I = -\sum_{n=0}^{\infty} \frac{\partial^2}{\partial b^2} \frac{x^{n+b-a+1}}{n+b-a+1} \Big|_0 \Big|_{b=0} = -\sum_{n=0}^{\infty} \frac{\partial^2}{\partial b^2} \frac{1}{n+b-a+1} \Big|_{b=0}$$

$$I = -\sum_{n=0}^{\infty} \frac{2}{(n+1-a)^3} = \psi^{(2)}(1-a)$$

$$I(a) = \psi^{(2)}(1-a)$$

$$\forall a < 1$$

Where: $\psi^{(2)}(z)$ is the Tetragamma function

Example: when $a = \frac{1}{2}$

$$I\left(\frac{1}{2}\right) = \psi^{(2)}\left(\frac{1}{2}\right) = -14\zeta(3)$$