

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_0^{\infty} \frac{a^x - b^x}{c^x - d^x} dx$$

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$$I = \int_0^{\infty} \frac{a^x - b^x}{c^x - d^x} dx, \quad I = \int_0^{\infty} \frac{\left(\frac{a}{c}\right)^x - \left(\frac{b}{c}\right)^x}{1 - \left(\frac{d}{c}\right)^x} dx = \int_0^{\infty} \frac{\alpha^x - \beta^x}{1 - \varphi^x} dx$$

$$\alpha, \beta, \varphi = \frac{a}{c}, \frac{b}{c}, \frac{d}{c} \text{ respectively}$$

$$I = \int_0^{\infty} (\alpha^x - \beta^x) \sum_{n=0}^{\infty} \varphi^{nx} dx \quad (\text{Geometric series: } |\varphi^x| < 1 \forall c > d)$$

$$I = \sum_{n=0}^{\infty} \int_0^{\infty} (\alpha^x - \beta^x) \varphi^{nx} dx$$

(Fubini – Tonelli Theorem justified by absolute convergence)

$$I = \sum_{n=0}^{\infty} \int_0^{\infty} (e^{\ln(\alpha)x} - e^{\ln(\beta)x}) e^{n \ln(\varphi)x} dx$$

$$I = \sum_{n=0}^{\infty} -\frac{1}{n \ln(\varphi) + \ln(\alpha)} + \frac{1}{n \ln(\varphi) + \ln(\beta)} \quad (\text{Laplace Transforms})$$

$$I = \frac{1}{\ln(\varphi)} \sum_{n=0}^{\infty} \frac{1}{n + \frac{\ln(\beta)}{\ln(\varphi)}} - \frac{1}{n + \frac{\ln(\alpha)}{\ln(\varphi)}}$$

$$I = \frac{1}{\ln(\varphi)} \sum_{n=0}^{\infty} -\left(\frac{1}{n+1} - \frac{1}{n + \frac{\ln(\beta)}{\ln(\varphi)}} \right) + \left(\frac{1}{n+1} - \frac{1}{n + \frac{\ln(\alpha)}{\ln(\varphi)}} \right)$$

Recall:

$$\Psi(z) + \gamma = \sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+z}$$

γ is the Euler – Mascheroni constant, $\Psi(z)$ is the digamma function

$$I = \frac{1}{\ln(\varphi)} \left(-\Psi\left(\frac{\ln(\beta)}{\ln(\varphi)}\right) - \gamma + \gamma + \Psi\left(\frac{\ln(\alpha)}{\ln(\varphi)}\right) \right)$$

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$$I = \frac{1}{\ln(\varphi)} \left(\psi \left(\frac{\ln(\alpha)}{\ln(\varphi)} \right) - \psi \left(\frac{\ln(\beta)}{\ln(\varphi)} \right) \right)$$

$$I = \frac{1}{\ln(\varphi)} \left(\psi(\log_{\varphi}(\alpha)) - \psi(\log_{\varphi}(\beta)) \right)$$

Where:

$$\alpha, \beta, \varphi = \frac{a}{c}, \frac{b}{c}, \frac{d}{c} \text{ respectively}$$

Valid:

$$\forall c > a, b, d$$