ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_{-1}^{1}\frac{dx}{\sqrt[5]{(1-x)^3(1+x)^2}}$$

Proposed by Vasile Mircea Popa-Romania

Solution 1 by Amin Hajiyev-Azerbaijan

$$I=\int_{-1}^{1}\frac{dx}{\sqrt[5]{(1-x)^3(1+x)^2}}\quad substutions\ x=2t-1\ \ dx=2dt$$

$$I=2\int_{0}^{1}\frac{dt}{\sqrt[5]{\left(1-(2t-1)\right)^3\left(1+(2t-1)\right)^2}}=2\int_{0}^{1}\frac{dt}{\sqrt[5]{(2-2t)^3 8t^2}}=$$

$$=\int_{0}^{1}\frac{dt}{\sqrt[5]{(1-t)^3t^2}}=\int_{0}^{1}(1-t)^{-\frac{3}{5}}t^{-\frac{2}{5}}dt=\int_{0}^{1}(1-t)^{\frac{2}{5}-1}t^{\frac{3}{5}-1}dt=$$

$$=\beta\left(\frac{2}{5};\frac{3}{5}\right)=\frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)}{\Gamma(1)}=\Gamma\left(1-\frac{3}{5}\right)\Gamma\left(\frac{3}{5}\right)=\frac{\pi}{\sin\left(\frac{3\pi}{5}\right)}=\frac{4\pi}{\sqrt{10+2\sqrt{5}}}$$

Note section:

- *The beta function* $\beta(x;y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_{0}^{1}(1-t)^{x-1}t^{y-1}dt\quad x,y>0$
- *Gamma reflection formula* $\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin(\pi z)}$
- $5\alpha=90^\circ\rightarrow\quad 2\alpha=90^\circ-3\alpha\rightarrow\sin(2\alpha)=\sin\left(\frac{\pi}{2}-3\alpha\right)$

$$\sin(2\alpha)=\cos(3\alpha)\rightarrow 2\sin(\alpha)\cos(\alpha)=4\cos^3(\alpha)-3\cos(\alpha)$$

$$4\sin^2(\alpha)+2\sin(\alpha)-1=0\rightarrow\sin\left(\frac{\pi}{10}\right)=\frac{\sqrt{5}-1}{4}$$

$$\cos\left(\frac{\pi}{10}\right)=\sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2}=\frac{\sqrt{10+2\sqrt{5}}}{4}=\cos\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)=\sin\left(\frac{3\pi}{5}\right)$$

Solution 2 by Tapas Das-India

$$\boldsymbol{Let\ x = 2t - 1\ then\ dx = 2dt}$$

$$\boldsymbol{when\ x = -1 \to t = 0, x = 1 \to t = 1}$$

$$\boldsymbol{using\ the\ above\ substitution\ we\ get}$$

$$\Omega = \int_0^1 \frac{2dt}{2^{\frac{3}{5}}(1-t)^{\frac{3}{5}}(2t)^{\frac{2}{5}}} = \int_0^1 t^{-\frac{2}{5}}(1-t)^{-\frac{3}{5}}dt = \int_0^1 t^{\frac{3}{5}-1}(1-t)^{\frac{2}{5}-1}dt$$

$$= \mathrm{B}\left(\frac{3}{5},\frac{2}{5}\right) = \frac{\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{2}{5}\right)}{\Gamma\left(\frac{2}{5}+\frac{3}{5}\right)} \overset{\Gamma(1)=1}{=} \Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{2}{5}\right) = \Gamma\left(\frac{3}{5}\right)\Gamma\left(1-\frac{3}{5}\right) = \frac{\pi}{\sin\left(\frac{3\pi}{5}\right)} = \frac{4\pi}{\sqrt{10+2\sqrt{5}}}$$

$$\boldsymbol{Note}: \sin\left(\frac{3\pi}{5}\right) = \sin 108^\circ = \cos 18^\circ = \frac{1}{4}\sqrt{10+2\sqrt{5}}$$