ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\infty} \frac{x \arctan(x)}{(1+x^2)(1+x)^2}\,dx$$

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$$I = \frac{1}{2}\int_0^{\infty}\left(\frac{1}{1+x^2}-\frac{1}{(1+x)^2}\right)\arctan(x)\,dx = \frac{1}{2}(I_1 - I_2)$$

$$I_2 = \int_0^{\infty}\frac{\arctan(x)}{1+x^2}\,dx \overset{\arctan(x)=u}{\cong} \int_0^{\frac{\pi}{2}} u\,du = \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}$$

$$I_2 = \int_0^{\infty}\frac{\arctan(x)}{(1+x)^2}\,dx \to^{IBP} \begin{array}{l} v = \int\left(\frac{1}{1+x}\right)^2 dx,\ v = -\frac{1}{1+x}. \\ u = \arctan(x), \frac{du}{dx} = \frac{1}{1+x^2} \end{array}$$

$$I_2 = \left[-\frac{\arctan(x)}{1+x}\right]_0^{\infty} + \int_0^{\infty}\frac{1}{(1+x)(1+x^2)}\,dx =$$

$$= \frac{1}{2}\left(\int_0^{\infty}\frac{1}{1+x^2}\,dx - \underbrace{\int_0^{\infty}\frac{x}{1+x^2}\,dx}_{x^2\to x} + \int_0^{\infty}\frac{1}{1+x}\,dx\right) =$$

$$= \left[\frac{1}{2}\arctan(x)\right]_0^{\infty} + \frac{1}{2}\lim_{x\to\infty}\ln\left(\frac{1+x}{\sqrt{1+x^2}}\right) - \frac{1}{2}\lim_{x\to 0}\ln\left(\frac{1+x}{\sqrt{1+x^2}}\right) =$$

$$= \frac{\pi}{4} + \frac{1}{2}\lim_{x\to\infty}\ln\left(\frac{\frac{1}{x}+1}{\sqrt{\frac{1}{x^2}+1}}\right) - \frac{1}{2}\lim_{x\to 0}\ln\left(\frac{\frac{1}{x}+1}{\sqrt{\frac{1}{x^2}+1}}\right) = \frac{\pi}{4}$$

$$\int_0^{\infty}\frac{x\arctan(x)}{(1+x^2)(1+x)^2}\,dx = \frac{1}{2}(I_1 - I_2) = \frac{1}{2}\left(\frac{\pi^2}{8}-\frac{\pi}{4}\right) = \frac{\pi^2}{16}-\frac{\pi}{8}$$