ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^\infty \frac{\ln^2(x)}{(x+1)^4}dx$$

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$$Let\ \ B(p,q)=\int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}dx\ \ p+q=n\to\ \ q=n-p$$

$$B(p,n-p)=\int_0^\infty \frac{x^{p-1}}{(1+x)^n}dx\to\ \ \Delta(n)=\int_0^\infty \frac{\ln^2(x)}{(x+1)^n}dx=$$

$$\lim_{p\to 1}\frac{\partial^2}{\partial p^2}B(p,n-p)$$

$$\Delta(n)=\frac{1}{\Gamma(n)}\left[\frac{\partial^2}{\partial p^2}\big(\Gamma(p)\Gamma(n-p)\big)\right]_{p=1}$$

$$Let:f(p)=\Gamma(p)\Gamma(n-p)$$

$$f'(p)=\Gamma(p)\psi^{(0)}(p)\Gamma(n-p)-\Gamma(p)\Gamma(n-p)\psi^{(0)}(n-p)$$

$$f''(p)=f'(p)\left(\left[\psi^{(0)}(p)-\psi^{(0)}(n-p)\right]^2+\left[\psi^{(1)}(p)+\psi^{(1)}(n-p)\right]\right)$$

$$In(1):\Delta(n)=\int_0^\infty \frac{\ln^2(x)}{(x+1)^n}dx=\frac{1}{n-1}\left[(H_{n-2})^2+\frac{\pi^2}{3}-H_{n-2}^{(2)}\right]$$

$$\Delta(4)=\frac{1}{3}\left[\left(\frac{3}{2}\right)^2+\frac{\pi^2}{3}-\frac{5}{4}\right]=\frac{1}{3}+\frac{\pi^2}{9}$$