ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\infty} \frac{x^2 \ln^2(x)}{x^4 + x^2 + 1} dx$$

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$$I = \int_0^{\infty} \frac{x^2 \ln^2(x)}{x^4 + x^2 + 1} dx \ \{x \to -x \quad x \le 0\ \} I = \int_{-\infty}^{0} \frac{x^2 (ln(-x))^2}{x^4 + x^2 + 1} dx =$$

$$= \int_{-\infty}^{0} \frac{x^2 (ln(x) - i\pi)^2}{x^4 + x^2 + 1} dx =$$

$$= \int_{-\infty}^{0} \frac{x^2 ln^2(x)}{x^4 + x^2 + 1} dx + 2i\pi \int_{-\infty}^{0} \frac{x^2 \ln(x)}{x^4 + x^2 + 1} dx - \pi^2 \int_{-\infty}^{0} \frac{x^2}{x^4 + x^2 + 1} dx$$

$$2I = \int_{-\infty}^{\infty} \frac{x^2 ln^2(x)}{x^4 + x^2 + 1} dx - 2i\pi \int_{-\infty}^{0} \frac{x^2 \ln(x)}{x^4 + x^2 + 1} dx - \pi^2 \int_0^{\infty} \frac{x^2}{x^4 + x^2 + 1} dx$$

$$2I = J - 2\pi i K - \pi^2 M$$

$$J = \int_{-\infty}^{\infty} \frac{x^2 ln^2(x)}{x^4 + x^2 + 1} dx, \qquad J(a) = \int_{-\infty}^{\infty} \frac{x^{2a}}{x^4 + x^2 + 1} dx, J = \frac{1}{4} lim_{a \to 1} \frac{d^2}{da^2} J(a)$$

$$J(a) = 2\pi i \sum Res$$

$$f(z) = \frac{z^{2a}}{z^4 + z^2 + 1} \to \ z^4 + z^2 + 1 = 0 \to \ z^2 = t$$

$$t^2 + t + 1 = 0 \to \ t_{1;2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} = \cos\left(\pm\frac{2\pi}{3}\right) + i\sin\left(\pm\frac{2\pi}{3}\right) = e^{\pm\frac{2\pi i}{3}}$$

$$z^2 = t \ \ z_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}, z_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, z_3 = \frac{1}{2} - \frac{i\sqrt{3}}{2}, z_4 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$\text{“the poles in the upper half} - \text{plane”}$$

$$z_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2} = e^{\frac{i\pi}{3}}, \quad z_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2} = e^{\frac{2i\pi}{3}}$$

$$Res(f, z_k) = \frac{z_k^{2a}}{\frac{d}{dz_k}\left(z_k^4 + z_k^2 + 1\right)} = \frac{z_k^{2a}}{4z_k^3 + 2z_k} = \frac{z_k^{2a-1}}{4z_k^2 + 2}$$

$$Res(f,z_1)=\frac{e^{\frac{i\pi(2a-1)}{3}}}{4e^{\frac{2i\pi}{3}}+2}=\frac{e^{\frac{i\pi(2a-1)}{3}}}{2i\sqrt{3}},Res(f,z_2)=\frac{e^{\frac{2i\pi(2a-1)}{3}}}{4e^{\frac{4i\pi}{3}}+2}=-\frac{e^{\frac{2i\pi(2a-1)}{3}}}{2i\sqrt{3}}$$

$$\sum Res=Res(f,z_1)+Res(f,z_2)=\frac{e^{\frac{i\pi(2a-1)}{3}}-e^{\frac{2i\pi(2a-1)}{3}}}{2i\sqrt{3}}$$

$$J(a)=\frac{\pi}{\sqrt{3}}\left(e^{\frac{i\pi}{3}(2a-1)}-e^{\frac{2i\pi}{3}(2a-1)}\right)\quad J=\frac{\pi}{4\sqrt{3}}lim_{a\to 1}\frac{d^2}{da^2}\left(e^{\frac{i\pi}{3}(2a-1)}-e^{\frac{2i\pi}{3}(2a-1)}\right)=$$

$$=\frac{\pi}{4\sqrt{3}}lim_{a\to 1}\left(\frac{16\pi^2}{9}e^{\frac{2i\pi}{3}(2a-1)}-\frac{4\pi^2}{9}e^{\frac{i\pi}{3}(2a-1)}\right)=\frac{\pi^3}{9\sqrt{3}}\left(4e^{\frac{2i\pi}{3}}-e^{\frac{i\pi}{3}}\right)=$$

$$=\frac{\pi^3}{9\sqrt{3}}\left(4cos\left(\frac{2\pi}{3}\right)+4isin\left(\frac{2\pi}{3}\right)-cos\left(\frac{\pi}{3}\right)-isin\left(\frac{\pi}{3}\right)\right)=-\frac{5\pi^3}{18\sqrt{3}}+\frac{i\pi^3}{6}$$

$$K=\int_{-\infty}^{0}\frac{x^2\ln(x)}{x^4+x^2+1}dx\ \ x\to -x\ \ K=\int_0^{\infty}\frac{x^2ln(x)+i\pi x^2}{1+x^2+x^4}dx$$

$$2K=\int_{-\infty}^{\infty}\frac{x^2\ln(x)}{1+x^2+x^4}dx+i\pi\int_0^{\infty}\frac{x^2}{x^4+x^2+1}dx=L+i\pi M$$

$$L=\int_{-\infty}^{\infty}\frac{x^2\ln(x)}{x^4+x^2+1}dx=\frac{1}{2}lim_{a\to 1}\frac{d}{da}J(a)=\frac{\pi}{2\sqrt{3}}lim_{a\to 1}\frac{d}{da}\left(e^{\frac{i\pi}{3}(2a-1)}-e^{\frac{2i\pi}{3}(2a-1)}\right)$$

$$=\frac{\pi}{2\sqrt{3}}\left(\frac{\pi}{\sqrt{3}}+i\pi\right)=\frac{\pi^2}{6}+\frac{\pi^2 i}{2\sqrt{3}}$$

$$M=\int_0^{\infty}\frac{x^2}{x^4+x^2+1}dx\ \ x\to -x\ \ M=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2+x^4}dx$$

$$M=\frac{1}{2}lim_{a\to 1}J(a)=\frac{\pi}{2\sqrt{3}}lim_{a\to 1}\left(e^{\frac{i\pi}{3}(2a-1)}-e^{\frac{2\pi i}{3}(2a-1)}\right)=$$

$$=\frac{\pi}{2\sqrt{3}}\left(cos\left(\frac{\pi}{3}\right)+isin\left(\frac{\pi}{3}\right)-cos\left(\frac{2\pi}{3}\right)-isin\left(\frac{2\pi}{3}\right)\right)=\frac{\pi}{2\sqrt{3}}$$

$$K=\frac{L}{2}+\frac{i\pi}{2}M=\frac{\pi^2}{12}+\frac{i\pi^2}{4\sqrt{3}}+\frac{i\pi^2}{4\sqrt{3}}=\frac{\pi^2}{12}+\frac{i\pi^2}{2\sqrt{3}}$$

$$2I=J-2\pi iK-\pi^2M=-\frac{5\pi^3}{18\sqrt{3}}+\frac{i\pi^3}{6}-\frac{i\pi^3}{6}+\frac{\pi^3}{\sqrt{3}}-\frac{\pi^3}{2\sqrt{3}}=\frac{2\pi^3}{9\sqrt{3}}$$

$$I=\int_0^{\infty}\frac{x^2ln^2(x)}{1+x^2+x^4}dx=\frac{\pi^3}{9\sqrt{3}}$$