ROMANIAN MATHEMATICAL MAGAZINE

Find :

$$\Delta = \int_0^\infty \frac{e^{-\pi x\phi}\sinh\left(\frac{\pi x}{\phi}\right)}{\sinh(\pi x)}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Shirvan Tahirov-Azerbaijan

$$\Delta = \int_0^\infty \frac{e^{-\pi x\phi}\sinh\left(\frac{\pi x}{\phi}\right)}{\sinh(\pi x)}dx = \int_0^\infty \frac{\frac{1}{2}\left(e^{-\pi x\phi}e^{\frac{\pi x}{\phi}} - e^{-\pi x\phi}e^{-\frac{\pi x}{\phi}}\right)}{\frac{1}{2}\left(\frac{1-e^{-2\pi x}}{e^{-\pi x}}\right)}dx =$$

$$= \int_0^\infty \frac{e^{-2\pi x} - e^{-2\pi\phi x}}{1-e^{-2\pi x}}dx = \sum_{j\geq 0}\int_0^\infty \left(e^{-2\pi(j+1)x} - e^{-2\pi(j+\phi)x}\right)dx =$$

$$= \sum_{j\geq 0}\left(\int_0^\infty e^{-2\pi(j+1)x}dx - \int_0^\infty e^{-2\pi(j+\phi)x}dx\right) =$$

$$= \sum_{j\geq 0}\left(\frac{1}{2\pi(j+1)} - \frac{1}{2\pi(j+\phi)}\right) = \frac{1}{2\pi}\sum_{j\geq 0}\left(\frac{1}{j+1} - \frac{1}{j+\phi}\right) = \frac{H_{\frac{1}{\phi}}}{2\pi}$$

Note :

$$\therefore \sinh(x) = \frac{e^x - e^{-x}}{2}, \qquad \therefore \int_0^\infty e^{-ax}dx = \frac{1}{a}, \qquad \therefore \frac{1}{1-e^{-2\pi x}} = \sum_{j\geq 0} e^{-2\pi jx}$$