ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \frac{1}{k}\left(\frac{2}{k^2}\big(K(k)-E(k)\big)-K(k)\right)dk = 1-\frac{\pi}{4}$$

Proposed by Shobhit Jain-India

Solution by Pratham Prasad-India

$$We\ Know:$$

$$\frac{d}{dk}K(k)=\frac{1}{k}\left(\frac{E(k)}{1-k^2}-K(k)\right)$$

$$\frac{d}{dk}E(k)=\frac{1}{k}(E(k)-K(k))$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}\right)=\frac{2}{k^3}(K(k)-E(k))+\frac{1}{k^2}\left(\frac{1}{k}(E(k)-K(k))-\frac{1}{k}\left(\frac{E(k)}{1-k^2}-K(k)\right)\right)$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}\right)=\frac{2}{k^3}(K(k)-E(k))-\frac{1}{k}\left(\frac{E(k)}{1-k^2}\right)$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}\right)=\frac{2}{k^3}(K(k)-E(k))-\left(\frac{1}{k}\left(\frac{E(k)}{1-k^2}\right)-\frac{1}{k}K(k)\right)-\frac{1}{k}K(k)$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}\right)=\frac{2}{k^3}(K(k)-E(k))-\frac{d}{dk}K(k)-\frac{1}{k}K(k)$$

$$\frac{d}{dk}\left(\frac{E(k)-K(k)}{k^2}+K(k)\right)=\frac{2}{k^3}(K(k)-E(k))-\frac{1}{k}K(k)$$

$$\left(\frac{E(k)-K(k)}{k^2}+K(k)\right)_0^1=\int_0^1\frac{1}{k}\left(\frac{2}{k^2}\big(K(k)-E(k)\big)-K(k)\right)dk$$

$$I=\int_0^1\frac{1}{k}\left(\frac{2}{k^2}\big(K(k)-E(k)\big)-K(k)\right)dk=E(1)-\lim_{k\to0}\left(\frac{E(k)-K(k)}{k^2}+K(k)\right)$$

$$\int_0^1 \frac{1}{k}\left(\frac{2}{k^2}\big(K(k)-E(k)\big)-K(k)\right)dk = 1-\lim_{k\to 0}\left(\frac{E(k)-K(k)}{k^2}\right) = 1-L-\frac{\pi}{2}$$

$$L=\lim_{k\to 0}\left(\frac{E(k)-K(k)}{k^2}\right)=\lim_{k\to 0}\left(\frac{E(k)-K(k)}{k^2}\right)=\lim_{k\to 0}\left(\frac{\frac{1}{k}\left(E(k)-K(k)-\frac{E(k)}{1-k^2}+K(k)\right)}{2k}\right)$$

$$L=-\lim_{k\to 0}\left(\frac{\big(E(k)\big)}{2(1-k^2)}\right)=\frac{-E(0)}{2}=-\frac{\pi}{4}$$

$$I=1-L-\frac{\pi}{2}=1-\frac{\pi}{4}$$

$$I=\int_0^1 \frac{1}{k}\left(\frac{2}{k^2}\big(K(k)-E(k)\big)-K(k)\right)dk=1-\frac{\pi}{4}$$