

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_0^1 \frac{\ln^2(x) \ln^4(1-x)}{x(1-x)} dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Nawar Alasadi-Iraq

$$I = \int_0^1 \frac{\ln^2(x) \ln^4(1-x)}{x} dx + \int_0^1 \frac{\ln^4(x) \ln^2(1-x)}{x} dx = I_1 + I_2$$

$$G(a, b) = \int_0^1 x^{a-1} (1-x)^b dx = \frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)} \Rightarrow \ln(aG(a, b)) \equiv S(a, b)$$

$$S(a, b) = \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} (a^k + b^k - (a+b)^k) \Rightarrow G(a, b) = \frac{1}{a} \exp(S(a, b))$$

$$I_1 = \frac{\partial^2}{\partial a^2} \frac{\partial^4}{\partial b^4} G(a, b)|_{a=b=0} = 48[a^3 b^4]e^S, \quad I_2 = \frac{\partial^4}{\partial a^4} \frac{\partial^2}{\partial b^2} G(a, b)|_{a=b=0} = 48[a^5 b^2]e^S$$

$$S(a, b) = -\zeta(2)ab + \zeta(3)ab(a+b) - \frac{\zeta(4)}{2}ab(2a^2 + 3ab + 2b^2) + \dots$$

$$e^S = [a^3 b^4] \left(S + \frac{S^2}{2} + \frac{S^3}{6} \right) = 5\zeta(7) - 2\zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4) + \frac{1}{2}\zeta(2)^2\zeta(3)$$

$$[a^5 b^2]e^S = [a^5 b^2] \left(S + \frac{S^2}{2} \right) = 3\zeta(7) - \zeta(2)\zeta(5) - \zeta(3)\zeta(4)$$

$$\begin{aligned} I &= 48([a^3 b^4]e^S + [a^5 b^2]e^S) \\ &= 48 \left(8\zeta(7) - 3\zeta(2)\zeta(5) - \frac{7}{2}\zeta(3)\zeta(4) + \frac{1}{2}\zeta(2)^2\zeta(3) \right) \\ &= 384\zeta(7) - 144\zeta(2)\zeta(5) - 168\zeta(3)\zeta(4) + 24\zeta(2)^2\zeta(3) \end{aligned}$$

Substitute $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$:

$$\begin{aligned} I &= 384\zeta(7) - 144 \left(\frac{\pi^2}{6} \right) \zeta(5) - 168 \left(\frac{\pi^4}{90} \right) \zeta(3) + 24 \left(\frac{\pi^4}{36} \right) \zeta(3) \\ &= 384\zeta(7) - 24\pi^2\zeta(5) - \frac{6}{5}\pi^4\zeta(3) \end{aligned}$$

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Solution 2 by Yang Silva Cartolin-Peru

$$I = \int_0^1 \frac{\log^2(x) \log^4(1-x)}{x} dx + \int_0^1 \frac{\log^2(x) \log^4(1-x)}{1-x} dx$$

$$I = \xi_1 + \xi_2$$

We will use the following generalization:

$$\log^\lambda(1-z) = (-1)^\lambda \lambda! \sum_{n=\lambda}^{\infty} \frac{[n]_\lambda}{n!} z^n = (-1)^\lambda \lambda! \sum_{n=1}^{\infty} \frac{H_n^{[\lambda-1]}}{n} z^n$$

Where $[n]_\lambda$ are the Stirling numbers of the first kind and H_n is the harmonic number

$$\stackrel{\lambda=4}{\Rightarrow} \log^4(1-z) = 4 \sum_{n=1}^{\infty} \frac{H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}}{n} z^n$$

$$\log^4(1-z) = 4 \sum_{n=1}^{\infty} \frac{\left(H_n - \frac{1}{n}\right)^3 - 3\left(H_n - \frac{1}{n}\right)\left(H_n^{(2)} - \frac{1}{n^2}\right) + 2\left(H_n^{(3)} - \frac{1}{n^3}\right)}{n} z^n$$

$$\log^4(1-z) = 4 \sum_{n=1}^{\infty} \frac{H_n^3 - \frac{3H_n^2}{n} + \frac{3H_n}{n^2} - \frac{1}{n^3}}{n} z^n - 12 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)} - \frac{H_n}{n^2} - \frac{H_n^{(2)}}{n} + \frac{1}{n^3}}{n} z^n + 8 \sum_{n=1}^{\infty} \frac{H_n^{(3)} - \frac{1}{n^3}}{n} z^n \therefore$$

Therefore:

$$\blacksquare \xi_1 = \int_0^1 \frac{\log^2(x) \log^4(1-x)}{x} dx$$

$$\xi_1 = 4 \sum_{n=1}^{\infty} \frac{H_n^3 - \frac{3H_n^2}{n} + \frac{3H_n}{n^2} - \frac{1}{n^3}}{n} \int_0^1 x^{n-1} \log^2(x) dx$$

$$- 12 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)} - \frac{H_n}{n^2} - \frac{H_n^{(2)}}{n} + \frac{1}{n^3}}{n} \int_0^1 x^{n-1} \log^2(x) dx + 8 \sum_{n=1}^{\infty} \frac{H_n^{(3)} - \frac{1}{n^3}}{n} \int_0^1 x^{n-1} \log^2(x) dx$$

$$\text{We know that: } \int_0^1 x^p \log^q(x) = \frac{(-1)^q q!}{(p+1)^{q+1}}$$

$$\xi_1 = 8 \sum_{n=1}^{\infty} \frac{H_n^3 - \frac{3H_n^2}{n} + \frac{3H_n}{n^2} - \frac{1}{n^3}}{n^4} - 24 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)} - \frac{H_n}{n^2} - \frac{H_n^{(2)}}{n} + \frac{1}{n^3}}{n^4} + 16 \sum_{n=1}^{\infty} \frac{H_n^{(3)} - \frac{1}{n^3}}{n^4}$$

$$\xi_1 = 8 \sum_{n=1}^{\infty} \frac{H_n^3}{n^4} - 24 \sum_{n=1}^{\infty} \frac{H_n^2}{n^5} + 48 \sum_{n=1}^{\infty} \frac{H_n}{n^6} - 48 \sum_{n=1}^{\infty} \frac{1}{n^7} - 24 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$$

$$+ 24 \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} + 16 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4}$$

Where:

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$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^4} = \frac{1}{240} (-34\pi^4\zeta(3) + 80\pi^2\zeta(5) + 3465\zeta(7))$$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^5} = 6\zeta(7) - \frac{\pi^2}{36} (\pi^2\zeta(3) + 6\zeta(5)) \wedge \sum_{n=1}^{\infty} \frac{H_n}{n^6} = -\frac{\pi^4}{90}\zeta(3) - \frac{\pi^2}{6}\zeta(5) + 4\zeta(7)$$

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} = \frac{1}{240} (2\pi^4\zeta(3) + 80\pi^2\zeta(5) - 765\zeta(7))$$

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} = \frac{\pi^4}{45}\zeta(3) + \frac{5\pi^2}{6}\zeta(5) - 10\zeta(7) \wedge \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4} = 18\zeta(7) - \frac{5\pi^2}{3}\zeta(5)$$

Therefore:

$$\xi_1 = \frac{1}{30} (-34\pi^4\zeta(3) + 80\pi^2\zeta(5) + 3465\zeta(7)) - 24 \left(6\zeta(7) - \frac{\pi^2}{36} (\pi^2\zeta(3) + 6\zeta(5)) \right)$$

$$+ 48 \left(-\frac{\pi^4}{90}\zeta(3) - \frac{\pi^2}{6}\zeta(5) + 4\zeta(7) \right) - 48\zeta(7) - \frac{1}{10} (2\pi^4\zeta(3) + 80\pi^2\zeta(5) - 765\zeta(7))$$

$$+ 24 \left(\frac{\pi^4}{45}\zeta(3) + \frac{5\pi^2}{6}\zeta(5) - 10\zeta(7) \right) + 16 \left(18\zeta(7) - \frac{5\pi^2}{3}\zeta(5) \right)$$

$$\xi_1 = -\frac{2\pi^4}{3}\zeta(3) - 16\pi^2\zeta(5) + 240\zeta(7) \therefore$$

$$\blacksquare \xi_2 = \int_0^1 \frac{\log^2(x) \log^4(1-x)}{1-x} dx \xrightarrow{\text{We do: } 1-x \rightarrow x} \xi_2 = \int_0^1 \frac{\log^2(1-x) \log^4(x)}{x} dx$$

Using the generalization mentioned above:

$$\stackrel{\lambda=2}{\Rightarrow} \log^2(1-z) = 2 \sum_{n=1}^{\infty} \frac{H_{n-1}}{n} z^n = 2 \sum_{n=1}^{\infty} \frac{H_n - \frac{1}{n}}{n} z^n$$

$$\text{Therefore: } \xi_2 = 2 \sum_{n=1}^{\infty} \frac{H_n - \frac{1}{n}}{n} \int_0^1 x^{n-1} \log^4(x) dx = 48 \left(\sum_{n=1}^{\infty} \frac{H_n}{n^6} - \sum_{n=1}^{\infty} \frac{1}{n^7} \right)$$

$$\xi_2 = 48 \left(-\frac{\pi^4}{90}\zeta(3) - \frac{\pi^2}{6}\zeta(5) + 3\zeta(7) \right)$$

$$\xi_2 = -\frac{8\pi^4}{15}\zeta(3) - 8\pi^2\zeta(5) + 144\zeta(7) \therefore$$

Thus:

$$I = \left(-\frac{2\pi^4}{3}\zeta(3) - 16\pi^2\zeta(5) + 240\zeta(7) \right) + \left(-\frac{8\pi^4}{15}\zeta(3) - 8\pi^2\zeta(5) + 144\zeta(7) \right)$$

$$I = -24\pi^2\zeta(5) + 384\zeta(7) - \frac{6\pi^4}{5}\zeta(3) \therefore$$