

ROMANIAN MATHEMATICAL MAGAZINE

Find :

$$\Delta = \int_0^1 \frac{\ln^2(1+x)\ln^2(1-x)}{1+x} dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$\# \text{Note: } a^2b^2 = \frac{1}{12} [(a+b)^4 + (a-b)^4 - 2a^4 - 2b^4]$$

$$I = \frac{1}{12} \left[\int_0^1 \frac{\log^4(1-x^2)}{1+x} dx + \int_0^1 \frac{\log^4\left(\frac{1-x}{1+x}\right)}{1+x} dx - 2 \int_0^1 \frac{\log^4(1-x)}{1+x} dx - 2 \int_0^1 \frac{\log^4(1+x)}{1+x} dx \right]$$

$$I = \frac{1}{12} (i_1 + i_2 - 2i_3 - 2i_4)$$

$$* i_4 = \int_0^1 \frac{\log^4(1+x)}{1+x} dx = \left[\frac{1}{5} \log^5(1+x) \right]_0^1 = \frac{1}{5} \log^5(2) \therefore$$

$$* i_3 = \int_0^1 \frac{\log^4(1-x)}{1+x} dx \xrightarrow{t=1-x \rightarrow dt=-dx} A_3 = \int_0^1 \frac{\log^4(t)}{2-t} dt = \frac{1}{2} \int_0^1 \frac{\log^4(t)}{1-\frac{t}{2}} dt$$

$$i_3 = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \int_0^1 t^{n-1} \log^4(t) dt$$

$$\# \text{Note: } \int_0^1 x^p \ln^q(x) dx = \frac{(-1)^q q!}{(p+1)^{q+1}}, i_3 = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{24}{n^5}\right) = 24 \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n^5} = 24 Li_5\left(\frac{1}{2}\right) \therefore$$

$$* i_2 = \int_0^1 \frac{\log^4\left(\frac{1-x}{1+x}\right)}{1+x} dx, \text{ let: } y = \frac{1-x}{1+x} \rightarrow dx = -\frac{2dy}{(1+y)^2}$$

$$i_2 = \int_0^1 \frac{\log^4(y)}{1+y} dy = \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^1 y^{n-1} \log^4(y) dy = 24 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5} = 24 \eta(5) \\ = \frac{45}{2} \zeta(5) \therefore$$

$$* i_1 = \int_0^1 \frac{\log^4(1-x^2)}{1+x} dx = \int_0^1 \frac{(1-x)\log^4(1-x^2)}{1-x^2} dx, \text{ let: } x = t^{\frac{1}{2}} \rightarrow dx = \frac{1}{2} t^{-\frac{1}{2}} dt$$

$$i_1 = \frac{1}{2} \int_0^1 \left(\frac{1-t^{\frac{1}{2}}}{t^{\frac{1}{2}}} \right) \frac{\log^4(1-t)}{1-t} dt, \text{ I.B.P: } \begin{cases} u = \frac{1-\sqrt{t}}{\sqrt{t}} \rightarrow du = -\frac{1}{2t^{\frac{3}{2}}} \\ dv = \frac{\log^4(1-t)}{1-t} dt \rightarrow v = -\frac{1}{5} \log^5(1-t) \end{cases}$$

$$i_1 = \frac{1}{2} \left(\left[-\frac{1}{5} \log^5(1-t) \left(\frac{1-\sqrt{t}}{\sqrt{t}} \right) \right]_0^1 - \frac{1}{10} \int_0^1 t^{-\frac{3}{2}} \log^5(1-t) dt \right)$$

$$i_1 = -\frac{1}{20} \int_0^1 t^{-\frac{3}{2}} \log^5(1-t) dt$$

$$\begin{aligned} \# \text{Note: } B(x, y) &= \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \rightarrow \frac{\partial}{\partial y} B(x, y) \\ &= \int_0^1 t^{x-1} (1-t)^{y-1} \log(1-t) dt \end{aligned}$$

$$i_1 = -\frac{1}{20} \lim_{(x,y) \rightarrow (-\frac{1}{2}, 1)} \frac{\partial}{\partial y^5} \left(\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \right) = \frac{\sqrt{\pi}}{10} \lim_{y \rightarrow 1} \frac{\partial}{\partial y^5} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{1}{2})} \right)$$

$$\text{Where: } \frac{\partial}{\partial y^5} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{1}{2})} \right) = \frac{\Gamma(y)}{\Gamma(y-\frac{1}{2})} \left(-\psi^{(0)}\left(y-\frac{1}{2}\right)^5 + 5\psi^{(0)}(y)\psi^{(0)}\left(y-\frac{1}{2}\right) \right)$$

$$-10 \frac{\Gamma(y)}{\Gamma(y-\frac{1}{2})} \left(\psi^{(0)}(y)^2 - \psi^{(1)}\left(y-\frac{1}{2}\right) + \psi^{(1)}(y) \right) \psi^{(0)}\left(y-\frac{1}{2}\right)^3$$

$$+10 \frac{\Gamma(y)}{\Gamma(y-\frac{1}{2})} \left(\psi^{(0)}(y)^3 - 3 \left(\psi^{(1)}\left(y-\frac{1}{2}\right) - \psi^{(1)}(y) \right) \psi^{(0)}(y) \right) \psi^{(0)}\left(y-\frac{1}{2}\right)^2$$

$$- \frac{\Gamma(y)}{\Gamma(y-\frac{1}{2})} \left(\psi^{(2)}\left(y-\frac{1}{2}\right) + \psi^{(2)}(y) \right) \psi^{(0)}\left(y-\frac{1}{2}\right)^2$$

$$-5 \frac{\Gamma(y)}{\Gamma(y-\frac{1}{2})} \left(\psi^{(0)}(y)^4 - 6 \left(\psi^{(1)}\left(y-\frac{1}{2}\right) - \psi^{(1)}(y) \right) \psi^{(0)}(y)^2 \right) \psi^{(0)}\left(y-\frac{1}{2}\right)$$

$$\begin{aligned}
 & +20 \frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \left(\psi^{(2)}\left(y-\frac{1}{2}\right) - \psi^{(2)}(y) \right) \psi^{(0)}(y) \psi^{(0)}\left(y-\frac{1}{2}\right) \\
 & -5 \frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \left(3\psi^{(1)}\left(y-\frac{1}{2}\right)^2 + 3\psi^{(1)}(y)^2 - 6\psi^{(1)}\left(y-\frac{1}{2}\right) \psi^{(1)}(y) \right) \psi^{(0)}\left(y-\frac{1}{2}\right) \\
 & +5 \frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \left(\psi^{(3)}\left(y-\frac{1}{2}\right) - \psi^{(3)}(y) \right) \psi^{(0)}\left(y-\frac{1}{2}\right) \\
 & + \frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \left(\psi^{(0)}(y)^5 - 10\psi^{(0)}(y)^3 \left(\psi^{(1)}\left(y-\frac{1}{2}\right) - \psi^{(1)}(y) \right) \right) \\
 & + \frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \left(10\psi^{(1)}\left(y-\frac{1}{2}\right) \psi^{(2)}\left(y-\frac{1}{2}\right) - 10\psi^{(1)}(y) \psi^{(2)}\left(y-\frac{1}{2}\right) \right) \\
 & - \frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \left(10\psi^{(0)}(y)^2 \left(\psi^{(2)}\left(y-\frac{1}{2}\right) - \psi^{(2)}(y) \right) + 10\psi^{(1)}\left(y-\frac{1}{2}\right) \psi^{(2)}(y) \right) \\
 & + \frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \left(10\psi^{(1)}(y) \psi^{(2)}(y) - \psi^{(4)}\left(y-\frac{1}{2}\right) + \psi^{(4)}(y) \right) \\
 & +5 \frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \psi^{(0)}(y) \left(3\psi^{(1)}\left(y-\frac{1}{2}\right)^2 - 6\psi^{(1)}(y) \psi^{(1)}\left(y-\frac{1}{2}\right) + 3\psi^{(1)}(y)^2 \right) \\
 & - \frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \psi^{(0)}(y) \left(\psi^{(3)}\left(y-\frac{1}{2}\right) - \psi^{(3)}(y) \right) \\
 & \lim_{y \rightarrow 1} \frac{\partial}{\partial y^5} \left(\frac{\Gamma(y)}{\Gamma\left(y-\frac{1}{2}\right)} \right) = \frac{-10\pi^2(12\zeta(3) + \log^3(4)) + 3(720\zeta(5) + 120\zeta(3)\log^2(4) + \log^5(4)) - 9\pi^4\log(4)}{3\sqrt{\pi}} \\
 & i_1 = -\frac{\pi^2}{3} (12\zeta(3) + \log^3(4)) + \frac{1}{10} (720\zeta(5) + 120\zeta(3)\log^2(4) + \log^5(4)) \\
 & \quad - \frac{3\pi^4}{10} \log(4)
 \end{aligned}$$

Therefore:

$$I = \frac{1}{12} \left(-\frac{\pi^2}{3} (12\zeta(3) + \log^3(4)) + \frac{1}{10} (720\zeta(5) + 120\zeta(3)\log^2(4) + \log^5(4)) - \frac{3\pi^4}{10} \log(4) \right) + \frac{1}{12} \left(\frac{45}{2} \zeta(5) \right) - \frac{1}{6} \left(24Li_5 \left(\frac{1}{2} \right) \right) - \frac{1}{6} \left(\frac{1}{5} \log^5(2) \right)$$

$$I = -4Li_5 \left(\frac{1}{2} \right) + \frac{63}{8} \zeta(5) + 4\zeta(3)\log^2(2) - \frac{\pi^2}{3} \zeta(3) + \frac{7}{30} \log^5(2) - \frac{2\pi^2}{9} \log^3(2) - \frac{9}{2} \log(2)\zeta(4)$$

Solution 2 by Nawar Alasadi-Iraq

Let: $a = \ln(1+x)$ and $\ln(1-x)$

$$\begin{cases} (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4b^3a + b^4 \\ (a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4b^3a + b^4 \\ (a+b)^4 + (a-b)^4 = 2a^4 + 12a^2b^2 + 2b^4 \end{cases}$$

$$\rightarrow a^2b^2 = \frac{1}{12} [(a+b)^4 + (a-b)^4 - 2a^4 - 2b^4]$$

$$\ln^2(1+x)\ln^2(1-x) =$$

$$\frac{1}{12} \left[\ln^4(1-x^2) + \ln^4 \left(\frac{1+x}{1-x} \right) - 2\ln^4(1+x) - 2\ln^4(1-x) \right]$$

Let: $I = \int_0^1 \frac{\ln^2(1+x)\ln^2(1-x)}{1+x} dx$

$$I = \frac{1}{12} I_1 + \frac{1}{12} I_2 - \frac{1}{12} I_3 - \frac{1}{6} I_4$$

Where:

$$\begin{cases} I_1 = \int_0^1 \frac{\ln^4(1-x^2)}{1+x} dx \\ I_2 = \int_0^1 \frac{\ln^4 \left(\frac{1+x}{1-x} \right)}{1+x} dx \\ I_3 = \int_0^1 \frac{\ln^4(1-x)}{1+x} dx \\ I_4 = \int_0^1 \frac{\ln^4(1+x)}{1+x} dx \end{cases}$$

Evaluating I_4 : Let $u = \ln(1+x) \rightarrow du = \frac{1}{1+x} dx$

$$I_4 = \int_0^{\ln(2)} u^4 du = \frac{\ln^5(2)}{5}$$

Evaluating I_3 : Let $u = 1 - x \rightarrow dx = -du, 1 + x = 2 - u$

$$I_3 = \int_0^1 \frac{\ln^4(u)}{2-u} du = \frac{1}{2} \int_0^1 \frac{\ln^4(u)}{1-\frac{u}{2}} du =$$

$$\frac{1}{2} \int_0^1 \ln^4(u) \sum_{n=0}^{\infty} \left(\frac{u}{2}\right)^n du = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int_0^1 u^n \ln^4(u) du =$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \frac{24}{(n+1)^5} = 24 \sum_{m=1}^{\infty} \frac{1}{2^m m^5} = 24 Li_5\left(\frac{1}{2}\right)$$

Evaluating I_2 : Let $y = \frac{1-x}{1+x} \rightarrow dx = \frac{-2}{(1+y)^2} dy, 1+x = \frac{2}{1+y}$

$$I_2 = \int_0^1 \frac{\ln^4(y)}{\frac{2}{1+y}} \frac{2}{(1+y)^2} dy = \int_0^1 \frac{\ln^4(y)}{1+y} dy =$$

$$- \sum_{n=0}^{\infty} (-1)^n \int_0^1 y^n \ln^4(y) dy = \sum_{n=0}^{\infty} (-1)^n \frac{24}{(1+n)^5} =$$

$$24 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5} = 24(1 - 2^{1-5})\zeta(5) = \frac{45}{2}\zeta(5)$$

Evaluating I_1 : Let $t = 1 - x^2 \rightarrow dx = -\frac{1}{2\sqrt{1-t}} dt, 1+x = \sqrt{1-t}$

$$I_1 = \int_0^1 \frac{\ln^4(t)}{1+\sqrt{1-t}} \frac{dt}{2\sqrt{1-t}} = \frac{1}{2} \int_0^1 \frac{1-\sqrt{1-t}}{t\sqrt{1-t}} \ln^4(t) dt =$$

$$\frac{1}{2} \int_0^1 \left(\frac{1}{\sqrt{1-t}} - 1 \right) \frac{\ln^4(t)}{t} dt$$

$$\text{Define } F(a) = \int_0^1 \left((1-t)^{-\frac{1}{2}} - 1 \right) t^{a-1} dt$$

$$F(a) = \frac{\Gamma(a)\sqrt{\pi}}{\Gamma\left(a+\frac{1}{2}\right)} - \frac{1}{a} = \frac{G(a)-1}{a}$$

$$\text{Where : } G(a) = \frac{\Gamma(1+a)\sqrt{\pi}}{\Gamma\left(a+\frac{1}{2}\right)} = \frac{2^{2a}\Gamma(1+a)^2}{\Gamma(1+2a)}$$

$$\ln G(a) = 2a\ln(2) + 2\ln\Gamma(1+a) - \ln\Gamma(1+2a) =$$

$$2(\ln 2)a - \zeta(2)a^5 + 2\zeta(2)a^3 - \frac{7}{2}\zeta(4)a^4 + 6\zeta(5)a^5 + o(a^6)$$

$$I_1 = \frac{1}{2}F^{(4)}(0) = \frac{1}{10}G^{(5)}(0) = 12[a^5]G(a)$$

$$\text{Let } S(a) = c_1a + c_2a^2 + c_3a^3 + c_4a^4 + c_5a^5$$

$$c_1 = 2\ln(2), c_2 = -\zeta(2), c_3 = 2\zeta(3), \quad c_4 = -\frac{7}{2}\zeta(4), c_5 = 6\zeta(5)$$

$$[a^5]e^{S(a)} = c_5 + c_1c_4 + c_2c_3 + \frac{1}{2}c_1^2c_3 + \frac{1}{2}c_2^2c_1 + \frac{1}{6}c_1^3c_2 + \frac{1}{120}c_1^5 =$$

$$6\zeta(5) - 7\zeta(4)\ln(2) - 2\zeta(2)\zeta(3) + 4\zeta(3)\ln^2(2) +$$

$$\frac{5}{2}\zeta(4)\ln(2) - \frac{4}{3}\zeta(2)\ln^3(2) + \frac{4}{15}\ln^5(2)$$

$$I_1 = 72\zeta(5) - 54\zeta(4)\ln(2) - 24\zeta(2)\zeta(3) +$$

$$48\zeta(3)\ln^2(2) - 16\zeta(2)\ln^3(2) + \frac{16}{5}\ln^5(2)$$

Finally, substituting all parts I:

$$I = \frac{1}{12}I_1 + \frac{1}{12}I_2 - \frac{1}{12}I_3 - \frac{1}{6}I_4$$

$$I = \left(6 + \frac{15}{8}\right)\zeta(5) - 4Li_5\left(\frac{1}{2}\right) - \frac{9}{2}\zeta(4)\ln(2) - 2\zeta(2)\zeta(3) +$$

$$4\zeta(3)\ln^2(2) - \frac{4}{3}\zeta(2)\ln^3(2) + \left(\frac{4}{15} - \frac{1}{30}\right)\ln^5(2)$$

$$\text{Using : } \zeta(2) = \frac{\pi^2}{6} \text{ and } \zeta(4) = \frac{\pi^4}{90}$$

$$I = \frac{63}{8}\zeta(5) - 4Li_5\left(\frac{1}{2}\right) - \frac{\pi^4}{20}\ln(2) - \frac{\pi^2}{3}\zeta(3) +$$

$$4\zeta(3)\ln^2(2) - \frac{2\pi^2}{9}\ln^3(2) + \frac{7}{30}\ln^5(2)$$