

Find :

$$\Omega = \int_0^1 x^2 \left(\arctan^3(x) + \ln \left(\frac{\ln(x)}{x^3 - 1} \right) \right) dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Nawar Alasadi-Iraq

$$\text{Let : } \Omega = \int_0^1 x^2 \left(\arctan^3(x) + \ln \left(\frac{\ln(x)}{x^3 - 1} \right) \right) dx =$$

$$\int_0^1 x^2 \arctan^3(x) dx + \int_0^1 x^2 \ln \left(\frac{-\ln(x)}{1 - x^3} \right) dx = I_1 + I_2$$

$$\text{For: } \int_0^1 x^2 \ln(-\ln(x)) dx \text{ let } x = e^{-t} \rightarrow dx = -e^{-t} dt$$

$$\int_0^\infty e^{-3t} \ln(t) dt \stackrel{3t=v}{=} \frac{1}{3} \int_0^\infty e^{-v} (\ln(v) - \ln(3)) dv = -\frac{\ln(3) + \gamma}{3}$$

$$\text{For: } \int_0^1 x^2 \ln(1 - x^3) dx \text{ let } v = x^3 \rightarrow dv = 3x^2 dx$$

$$\frac{1}{3} \int_0^1 \ln(1 - v) dv = \frac{1}{3} [-(1 - v) \ln(1 - v) - v]_0^1 = -\frac{1}{3}$$

$$I_2 = -\frac{\ln(3) + \gamma}{3} + \frac{1}{3} = \frac{1 - \gamma - \ln(3)}{3}$$

$$I_1 = \int_0^1 x^2 \arctan^3(x) dx \quad (I.B.P \ (u = \arctan^3(x), dv = x^2 dx))$$

$$I_1 = \left[\frac{x^3}{3} \arctan^3(x) \right]_0^1 - \int_0^1 \frac{x^3}{1 + x^2} \arctan^2(x) dx =$$

$$\frac{\pi^3}{192} - \int_0^1 \left(x - \frac{x}{1 + x^2} \right) \arctan^2(x) dx = \frac{\pi^3}{192} - J_1 + J_2$$

$$J_1 = \int_0^1 x \arctan^2(x) dx \quad (I.B.P \ (u = \arctan^2(x), dv = x dx))$$

$$J_1 = \left[\frac{x^2}{2} \arctan^2(x) \right]_0^1 - \int_0^1 \frac{x^2}{1 + x^2} \arctan(x) dx =$$

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$$\frac{\pi^2}{32} - \int_0^1 \left(1 - \frac{1}{1+x^2}\right) \arctan(x) dx = \frac{\pi^2}{32} - \left(\frac{\pi}{4} - \frac{1}{2}\ln(2)\right) + \frac{\pi^2}{32} = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln(2)$$

$$J_2 = \int_0^1 \frac{x}{1+x^2} \arctan^2(x) dx \{x = \tan(u)\} \rightarrow \int_0^{\frac{\pi}{4}} u^2 \tan(u) du =$$

$$[-u^2 \ln(\cos(u))] \frac{\pi}{4} + 2 \int_0^{\frac{\pi}{4}} u \ln(\cos(u)) du = \frac{\pi^2}{32} \ln(2) + 2 \int_0^{\frac{\pi}{4}} u \ln(\cos(u)) du$$

$$\text{Using } \ln(\cos(u)) = -\ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos(2nu)$$

$$2 \int_0^{\frac{\pi}{4}} u \ln(\cos(u)) du = -\frac{\pi^2}{16} \ln(2) + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^{\frac{\pi}{4}} u \cos(2nu) du =$$

$$-\frac{\pi^2}{16} \ln(2) + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\pi}{8} \sin\left(\frac{\pi n}{2}\right) + \frac{\cos\left(\frac{\pi n}{2}\right) - 1}{4n^2} \right) =$$

$$-\frac{\pi^2}{16} \ln(2) + \frac{\pi}{4} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{n-1} (\cos(\frac{\pi n}{2}) - 1)}{n^3} =$$

$$-\frac{\pi^2}{16} \ln(2) + \frac{\pi}{4} G + \frac{1}{2} \left(-\frac{7}{8} \zeta(3) - \frac{1}{8} \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{k^3} \right) =$$

$$-\frac{\pi^2}{16} \ln(2) + \frac{\pi}{4} G + \frac{1}{2} \left(-\frac{7}{8} \zeta(3) + \frac{7}{32} \zeta(3) \right) = -\frac{\pi^2}{16} \ln(2) + \frac{\pi}{4} G - \frac{21}{64} \zeta(3)$$

$$J_2 = \frac{\pi^2}{32} \ln(2) - \frac{\pi^2}{16} \ln(2) + \frac{\pi}{4} G - \frac{21}{64} \zeta(3)$$

$$J_2 = -\frac{\pi^2}{32} \ln(2) + \frac{\pi}{4} G - \frac{21}{64} \zeta(3)$$

$$I_1 = \frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} - \frac{1}{2} \ln(2) - \frac{\pi^2}{32} \ln(2) + \frac{\pi}{4} G - \frac{21}{64} \zeta(3)$$

$$I_2 = \frac{1 - \gamma - \ln(3)}{3}$$

$$\diamond = \frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} - \left(\frac{\pi^2 + 16}{32} \right) \ln(2) + \frac{\pi}{4} G - \frac{21}{64} \zeta(3) + \frac{1 - \gamma - \ln(3)}{3}$$

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Solution 2 by Yang Silva Cartolin-Peru

$$\Omega = \int_0^1 x^2 \left(\arctan^3(x) + \ln \left(\frac{\ln(x)}{x^3 - 1} \right) \right) dx$$

$$\Omega = \int_0^1 x^2 \arctan^3(x) dx + \int_0^1 x^2 \log(-\log(x)) dx - \int_0^1 x^2 \log(1 - x^3) dx$$

$$I = \xi_1 + \xi_2 - \xi_3$$

$$\blacksquare \xi_1 = \int_0^1 x^2 \arctan^3(x) dx$$

$$\text{We do: } t = \arctan(x) \rightarrow x = \tan(t) \rightarrow dx = \sec^2(t) dt$$

$$x = 1 \rightarrow t = \frac{\pi}{4} \wedge x = 0 \rightarrow t = 0 \Rightarrow \xi_1 = \int_0^{\frac{\pi}{4}} t^3 \tan^2(t) \sec^2(t) dt$$

$$I.B.P: \begin{cases} u = t^3 \rightarrow du = 3t^2 dt \\ dv = \tan^2(t) \sec^2(t) dt \rightarrow v = \frac{1}{3} \tan^3(t) \end{cases}$$

$$\Rightarrow \xi_1 = \left[\frac{t^3}{3} \tan^3(t) \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} t^2 \tan^3(t) dt = \frac{\pi^3}{192} - i_1$$

$$\gg i_1 = \int_0^{\frac{\pi}{4}} t^2 \tan^3(t) dt \xrightarrow{\tan^2(x) = \sec^2(x) - 1} i_1 = \int_0^{\frac{\pi}{4}} t^2 \tan(t) \sec^2(t) dt - \int_0^{\frac{\pi}{4}} t^2 \tan(t) dt$$

$$i_1 = a - b \gg a = \int_0^{\frac{\pi}{4}} t^2 \tan(t) \sec^2(t) dt$$

$$I.B.P: \begin{cases} u = t^2 \rightarrow du = 2t dt \\ dv = \tan(t) \sec^2(t) dt \rightarrow v = \frac{1}{2} \tan^2(t) \end{cases}$$

$$\Rightarrow a = \left[\frac{t^2}{2} \tan^2(t) \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} t \tan^2(t) dt = \frac{\pi^2}{32} - \int_0^{\frac{\pi}{4}} t \sec^2(t) dt + \int_0^{\frac{\pi}{4}} t dt$$

$$a = \frac{\pi^2}{16} - \int_0^{\frac{\pi}{4}} t \sec^2(t) dt$$

$$I. B. P: \begin{cases} u = t \rightarrow du = dt \\ dv = \sec^2(t) dt \rightarrow v = \tan(t) \end{cases} \Rightarrow a = \frac{\pi^2}{16} - [t \tan(t)]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan(t) dt$$

$$a = \frac{\pi^2}{16} - \frac{\pi}{4} + [-\log(\cos(t))]_0^{\frac{\pi}{4}} = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \log(2) \therefore \gg b = \int_0^{\frac{\pi}{4}} t^2 \tan(t) dt$$

$$I. B. P: \begin{cases} u = t^2 \rightarrow du = 2t dt \\ dv = \tan(t) \rightarrow v = -\log(\cos(t)) \end{cases}$$

$$\Rightarrow b = [-t^2 \log(\cos(t))]_0^{\frac{\pi}{4}} + 2 \int_0^{\frac{\pi}{4}} t \log(\cos(t)) dt = \frac{\pi^2}{32} \log(2) + 2 \int_0^{\frac{\pi}{4}} t \log(\cos(t)) dt$$

$$\text{Fourier Series: } \log(\cos(x)) = -\log(2) - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \cos(2kx)$$

$$\Rightarrow b = \frac{\pi^2}{32} \log(2) - 2 \log(2) \int_0^{\frac{\pi}{4}} t dt - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \int_0^{\frac{\pi}{4}} t \cos(2kt) dt$$

$$b = -\frac{\pi^2}{32} \log(2) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left[\frac{\pi}{8k} \sin\left(\frac{\pi k}{2}\right) + \frac{1}{4k^2} \cos\left(\frac{\pi k}{2}\right) - \frac{1}{4k^2} \right]$$

$$b = -\frac{\pi^2}{32} \log(2) - \frac{\pi}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin\left(\frac{\pi k}{2}\right) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \cos\left(\frac{\pi k}{2}\right) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3}$$

We do:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin\left(\frac{\pi k}{2}\right) \xrightarrow{k=2n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \cos\left(\frac{\pi k}{2}\right) \xrightarrow{k=2n} - \frac{1}{8} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3}$$

$$\Rightarrow b = -\frac{\pi^2}{32} \log(2) + \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} - \frac{7}{16} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3}$$

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$$\text{Dirichlet Beta Function: } \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \beta(2) = G$$

$$\text{Dirichlet Beta Function: } \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \eta(3) = \frac{3}{4}\zeta(3) \Rightarrow b = -\frac{\pi^2}{32}\log(2) + \frac{\pi}{4}G - \frac{21}{64}\zeta(3) \therefore$$

Therefore:

$$i_1 = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log(2) - \left(-\frac{\pi^2}{32}\log(2) + \frac{\pi}{4}G - \frac{21}{64}\zeta(3) \right)$$

$$i_1 = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log(2) + \frac{\pi^2}{32}\log(2) - \frac{\pi}{4}G + \frac{21}{64}\zeta(3) \therefore$$

This:

$$\xi_1 = \frac{\pi^3}{192} - \left(\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log(2) + \frac{\pi^2}{32}\log(2) - \frac{\pi}{4}G + \frac{21}{64}\zeta(3) \right)$$

$$\xi_1 = \frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} - \frac{1}{2}\log(2) - \frac{\pi^2}{32}\log(2) + \frac{\pi}{4}G - \frac{21}{64}\zeta(3) \therefore$$

$$\blacksquare \xi_2 = \int_0^1 x^2 \log(-\log(x)) dx$$

$$\text{We do: } t = -\log(x) \rightarrow x = e^{-t} \rightarrow dx = -e^{-t}dt$$

$$x = 1 \rightarrow t = 0 \wedge x = 0 \rightarrow t = \infty \Rightarrow \xi_2 = \int_0^{\infty} e^{-3t} \log(t) dt$$

$$\text{We do: } x = 3t \rightarrow dt = \frac{dx}{3}$$

$$\Rightarrow \xi_2 = \frac{1}{3} \int_0^{\infty} e^{-x} \log\left(\frac{x}{3}\right) dx = \frac{1}{3} \left(\int_0^{\infty} e^{-x} \log(x) dx - \log(3) \int_0^{\infty} e^{-x} dx \right)$$

$$\xi_2 = \frac{1}{3} \left(\int_0^{\infty} e^{-x} \log(x) dx - \log(3) \right)$$

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$$\text{Gamma Function: } \Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx \rightarrow \Gamma'(z) = \int_0^{\infty} x^{z-1} \log(x) e^{-x} dx$$

$$\xi_2 = \frac{1}{3} (\Gamma'(1) - \log(3)) = -\frac{\gamma}{3} - \frac{1}{3} \log(3) \therefore$$

$$\blacksquare \xi_3 = \int_0^1 x^2 \log(1-x^3) dx = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{3n+2} dx = -\frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = -\frac{1}{3} \therefore$$

Thus:

$$I = \frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} - \frac{1}{2} \log(2) - \frac{\pi^2}{32} \log(2) + \frac{\pi}{4} G - \frac{21}{64} \zeta(3) - \frac{\gamma}{3} - \frac{1}{3} \log(3) + \frac{1}{3}$$

Solution 3 by Lara Gangler-Hungary

$$I = \int_0^1 x^2 \left((\tan^{-1} x)^3 + \ln \left(\frac{\ln(x)}{x^3 - 1} \right) \right) dx$$

$$I = \int_0^1 x^2 (\tan^{-1} x)^3 dx + \int_0^1 x^2 \ln \left(\frac{\ln(x)}{x^3 - 1} \right) dx = I_1 + I_2$$

Calculating I_1 Integration by parts - 1:

$$\left(\frac{x^3}{3} (\tan^{-1} x)^3 \right)' = x^2 (\tan^{-1} x)^3 + \frac{x^3}{3} 3 (\tan^{-1} x)^2 \frac{1}{1+x^2}$$

$$\frac{x^3}{3} (\tan^{-1} x)^3 \Big|_0^1 = I_1 + \int_0^1 x^3 (\tan^{-1} x)^2 \frac{dx}{1+x^2}$$

$$\begin{aligned} \frac{\pi^3}{192} &= I_1 + \int_0^1 x^3 (\tan^{-1} x)^2 \frac{dx}{1+x^2} = I_1 + \int_0^1 (\tan^{-1} x)^2 \frac{(x^3 + x - x) dx}{1+x^2} \\ &= I_1 + \int_0^1 \left(x (\tan^{-1} x)^2 - \frac{x (\tan^{-1} x)^2}{1+x^2} \right) dx = I_1 + J_{11} - J_{12} \end{aligned}$$

Calculating J_{11} Integration by parts - 2:

$$\left(\frac{x^2}{2} (\tan^{-1} x)^2 \right)' = x (\tan^{-1} x)^2 + \frac{x^2}{2} 2 \tan^{-1} x \frac{1}{1+x^2}$$

$$\begin{aligned}
 \frac{x^2}{2} (\tan^{-1} x)^2 \Big|_0^1 &= J_{11} + \int_0^1 x^2 \tan^{-1} x \frac{dx}{1+x^2} = J_{11} + \int_0^1 x^2 \tan^{-1} x \frac{(x^2+1-1)}{1+x^2} dx = \\
 &= J_{11} + \int_0^1 \left(\tan^{-1} x - \frac{\tan^{-1} x}{1+x^2} \right) dx \\
 &= J_{11} + \left[\left(x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) \right) - \frac{1}{2} (\tan^{-1} x)^2 \right] \Big|_0^1 \\
 \frac{\pi^2}{32} &= J_{11} + \frac{\pi}{4} - \frac{1}{2} \ln(2) - \frac{\pi^2}{32} \rightarrow J_{11} = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \ln(2)
 \end{aligned}$$

Calculating J_{12} Integration by parts - 3:

$$\begin{aligned}
 \left(\frac{1}{2} \ln(1+x^2) (\tan^{-1} x)^2 \right)' &= \frac{x(\tan^{-1} x)^2}{1+x^2} + \frac{1}{2} \ln(1+x^2) 2 \frac{\tan^{-1} x}{1+x^2} \\
 \left(\frac{1}{2} \ln(1+x^2) (\tan^{-1} x)^2 \right) \Big|_0^1 &= J_{12} + \int_0^1 \ln(1+x^2) \frac{\tan^{-1} x}{1+x^2} dx = J_{12} + K_{12} \\
 \frac{\pi^2}{32} \ln(2) &= J_{12} + K_{12}
 \end{aligned}$$

Calculating K_{12}

$$\begin{aligned}
 K_{12} &= \int_0^1 \ln(1+x^2) \frac{\tan^{-1} x}{1+x^2} dx \\
 x = \tan y &\rightarrow dx = \sec^2 y dy \text{ \& } 1+x^2 = \sec^2 y \\
 K_{12} &= \int_0^{\pi/4} \ln(\sec^2 y) \frac{y}{\sec^2 y} \sec^2 y dy = - \int_0^{\pi/4} 2y \ln(\cos y) dy = -2 L_{12}
 \end{aligned}$$

$$\begin{aligned}
 \text{Calculating } L_{12} &= \int_0^{\pi/4} y \ln(\cos y) dy = \int_0^{\pi/4} y \left(-\ln 2 + \sum_{k=1}^{\infty} \frac{\cos 2ky}{k} (-1)^{k+1} \right) dy = \\
 &= -\frac{1}{2} \left(\frac{\pi}{4} \right)^2 \ln 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int_0^{\pi/4} y \cos 2ky dy = -\frac{\pi^2}{32} \ln(2) + \\
 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{2ky \sin 2ky + \cos 2ky}{(2k)^2} \Big|_0^{\pi/4} &= -\frac{\pi^2}{32} \ln(2) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k^3} \left(\frac{k\pi}{2} \sin \frac{k\pi}{2} + \cos \frac{k\pi}{2} - 1 \right) = \\
 &= -\frac{\pi^2}{32} \ln(2) - \frac{1}{4} \times \frac{3}{4} \zeta(3) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k^3} \left(\frac{k\pi}{2} \sin \frac{k\pi}{2} + \cos \frac{k\pi}{2} \right) = -\frac{\pi^2}{32} \ln(2) - \frac{3\zeta(3)}{16} + \\
 \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{4(2k)^3} (k\pi \sin k\pi + \cos k\pi) &+ \sum_{k=1}^{\infty} \frac{(-1)^{2k}}{4(2k-1)^3} \left(\frac{(2k-1)\pi}{2} \sin \frac{(2k-1)\pi}{2} + \cos \frac{(2k-1)\pi}{2} \right) = \\
 &= -\frac{\pi^2}{32} \ln(2) - \frac{3\zeta(3)}{16} + \sum_{k=1}^{\infty} \left(\frac{(-1)^{2k+1}}{4(2k)^3} \cos k\pi + \frac{(-1)^{2k}}{4(2k-1)^3} \frac{(2k-1)\pi}{2} \sin \frac{(2k-1)\pi}{2} \right) = \\
 &= -\frac{\pi^2}{32} \ln(2) - \frac{3\zeta(3)}{16} + \sum_{k=1}^{\infty} \left(\frac{-1}{32k^3} (-1)^k + \frac{\pi}{8(2k-1)^2} (-1)^{k-1} \right) = \\
 &= -\frac{\pi^2}{32} \ln(2) - \frac{3\zeta(3)}{16} + \frac{1}{32} \frac{3\zeta(3)}{4} + \frac{\pi G}{8} = \frac{\pi G}{8} - \frac{\pi^2}{32} \ln(2) - \frac{21\zeta(3)}{128}
 \end{aligned}$$

$$\rightarrow K_{12} = \int_0^1 \ln(1+x^2) \frac{\tan^{-1} x}{1+x^2} dx = \frac{\pi^2}{16} \ln(2) + \frac{21\zeta(3)}{64} - \frac{\pi G}{4}$$

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$$\frac{\pi^2}{32} \ln(2) = J_{12} + K_{12}$$

$$\rightarrow J_{12} = \int_0^1 \frac{x(\tan^{-1} x)^2}{1+x^2} dx = \frac{\pi G}{4} - \frac{\pi^2}{32} \ln(2) - \frac{21\zeta(3)}{64}$$

$$\frac{\pi^3}{192} = I_1 + J_{11} - J_{12}$$

$$I_1 = \frac{\pi^3}{192} + \frac{\pi G}{4} - \frac{\pi^2}{32} \ln(2) - \frac{21\zeta(3)}{64} - \left(\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \ln(2) \right)$$

$$\rightarrow I_1 = \int_0^1 x^2 (\tan^{-1} x)^3 = \frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} - \frac{\pi^2}{32} \ln(2) + \frac{\pi G}{4} - \frac{1}{2} \ln(2) - \frac{21\zeta(3)}{64}$$

Calculating I_2 : $I_2 = \int_0^1 x^2 \ln\left(\frac{\ln(x)}{x^3-1}\right) dx = \int_0^1 x^2 \ln\left(\frac{-\ln(x)}{1-x^3}\right) dx = I_{21} - I_{22}$

Calculating I_{21} : $u = -\ln(x) \rightarrow x = e^{-u} \rightarrow dx = -e^{-u} du$

$$I_{21} = \int_0^1 x^2 \ln(-\ln(x)) dx = \int_0^\infty e^{-2u} \ln(u) e^{-u} du = \int_0^\infty e^{-3u} \ln(u) du$$

$$w = 3u \rightarrow du = \frac{dw}{3}$$

$$I_{21} = \int_0^\infty e^{-3u} \ln(u) du = \frac{1}{3} \int_0^\infty e^{-w} \ln\left(\frac{w}{3}\right) dw = \frac{1}{3} \int_0^\infty e^{-w} (\ln(w) - \ln(3)) dw$$

$$= -\frac{\gamma + \ln(3)}{3}$$

Calculating I_{22}

$$I_{22} = \int_0^1 x^2 \ln(1-x^3) dx = \int_0^1 -x^2 \sum_{k=1}^\infty \frac{x^{3k}}{k} dx = \sum_{k=1}^\infty -\frac{1}{k} \int_0^1 x^{3k+2} dx = \sum_{k=1}^\infty -\frac{1}{k} \frac{1}{3k+3}$$

$$= -\frac{1}{3} \sum_{k=1}^\infty \frac{1}{k(k+1)} = -\frac{1}{3} \sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1}{k+1} \right) = -\frac{1}{3}$$

$$I_2 = I_{21} - I_{22} = -\frac{\gamma + \ln(3)}{3} + \frac{1}{3} = \frac{1 - \gamma - \ln(3)}{3}$$

$$I = I_1 + I_2 = \frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} - \frac{\pi^2}{32} \ln(2) + \frac{\pi G}{4} - \frac{1}{2} \ln(2) + \frac{\pi G}{4} - \frac{21\zeta(3)}{64} + \frac{1 - \gamma - \ln(3)}{3}$$

$$\rightarrow I = \int_0^1 x^2 \left((\tan^{-1} x)^3 + \ln\left(\frac{\ln(x)}{x^3-1}\right) \right) dx =$$

$$I = \frac{1}{3} + \frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} - \frac{\pi^2}{32} \ln(2) + \frac{\pi G}{4} - \frac{1}{2} \ln(2) - \frac{1}{3} \ln(3) - \frac{21\zeta(3)}{64} - \frac{\gamma}{3}$$

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Solution 4 by Samuel Ajibade-Nigeria

$$I = \int_0^1 x^2 \left(\arctan^3(x) + \ln \left(\frac{\ln(x)}{x^3 - 1} \right) \right) dx$$

$$I = \int_0^1 x^2 \arctan^3(x) dx + \int_0^1 x^2 \ln \left(\frac{\ln(x)}{x^3 - 1} \right) dx$$

$$: I = J + K \quad \dots \text{equation 1, } J = \int_0^1 x^2 \arctan^3(x) dx$$

$$\text{Using IBP: } u = \arctan^3(x), dv = x^2, \quad du = \frac{3 \arctan^2(x)}{1 + x^2} dx, v = \frac{x^3}{3}$$

$$J = \frac{x^3}{3} \arctan^3(x) \Big|_0^1 - \int_0^1 \frac{x^3}{1 + x^2} \arctan^2(x) dx$$

$$J = \frac{\pi^3}{192} - \int_0^1 \left(x - \frac{x}{1 + x^2} \right) \arctan^2(x) dx$$

$$J = \frac{\pi^3}{192} - \int_0^1 x \arctan^2(x) dx + \int_0^1 \frac{x \arctan^2(x)}{1 + x^2} dx$$

$$\text{Let } J = \frac{\pi^3}{192} - J_1 + J_2 \quad \dots \text{equation 2, } J_1 = \int_0^1 x \arctan^2(x) dx$$

$$\text{IBP: } u = \arctan^2(x), du = \frac{2 \arctan(x)}{1 + x^2}, v = \frac{x^2}{2}$$

$$J_1 = \frac{x^2}{2} \arctan^2(x) \Big|_0^1 - \int_0^1 \frac{x^2 \arctan(x)}{1 + x^2} dx$$

$$J_1 = \frac{\pi^2}{32} - \int_0^1 \arctan(x) dx + \int_0^1 \frac{\arctan(x)}{1 + x^2} dx$$

$$J_1 = \frac{\pi^2}{32} - \left(x \arctan(x) - \frac{1}{2} \ln(1 + x^2) \right) \Big|_0^1 + \frac{1}{2} \arctan^2(x) \Big|_0^1$$

$$J_1 = \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2} \ln(2) + \frac{\pi^2}{32}$$

$$J_1 = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \ln(2)$$

$$J_2 = \int_0^1 \frac{x \arctan^2(x)}{1 + x^2} dx$$

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$$\text{Let } t = \arctan(x), dt = \frac{dx}{1+x^2}, t \in \left[0, \frac{\pi}{4}\right]$$

$$J_2 = \int_0^{\frac{\pi}{4}} t^2 \tan(t) dt$$

$$\text{By IBP: } u = t^2, du = 2t dt, v = -\ln(\cos(t))$$

$$J_2 = -t^2 \ln(\cos(t)) \Big|_0^{\frac{\pi}{4}} + 2 \int_0^{\frac{\pi}{4}} t \ln(\cos(t)) dt$$

By Fourier expansion of $\ln(\cos(t))$

$$J_2 = \frac{\pi^2}{32} \ln(2) - 2 \int_0^{\frac{\pi}{4}} t \left(\ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos(2nt) \right) dt$$

$$J_2 = \frac{\pi^2}{32} \ln(2) - 2 \ln(2) \int_0^{\frac{\pi}{4}} t dt - 2 \int_0^{\frac{\pi}{4}} t \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos(2nt) dt$$

Fubini – Tonelli Theorem

$$J_2 = \frac{\pi^2}{32} \ln(2) - \ln(2) \frac{\pi^2}{16} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^{\frac{\pi}{4}} t \cos(2nt) dt$$

$$\text{By IBP: } u = t, v = \frac{\sin(2nt)}{2n} \left\{ \begin{array}{l} \text{NB: } \sin\left(\frac{\pi}{2}n\right) = \begin{cases} 0 & n \text{ is even} \\ \pm 1 & n \text{ is odd} \end{cases} \\ \therefore \sin\left(\frac{\pi}{2}(2k+1)\right) = (-1)^k \\ \cos\left(\frac{\pi}{2}n\right) = \begin{cases} 0 & n \text{ is odd} \\ \pm 1 & n \text{ is even} \end{cases} \\ \therefore \cos\left(\frac{\pi}{2}(2k)\right) = (-1)^k \end{array} \right.$$

$$J_2 = -\frac{\pi^2}{32} \ln(2) - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(t \frac{\sin(2nt)}{2n} + \frac{\cos(2nt)}{4n^2} \right) \Big|_0^{\frac{\pi}{4}}$$

$$J_2 = -\frac{\pi^2}{32} \ln(2) - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{\pi \sin\left(\frac{\pi}{2}n\right)}{8n} + \frac{\cos\left(\frac{\pi}{2}n\right) - 1}{4n^2} \right)$$

$$J_2 = -\frac{\pi^2}{32} \ln(2) - \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin\left(\frac{\pi}{2}n\right) - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \cos\left(\frac{\pi}{2}n\right) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

For the first series, let $n = 2k + 1$ (the odd integer sub)

For the second series, let $n = 2m$ (the even integer sub): $n \geq 2$

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$$J_2 = -\frac{\pi^2}{32} \ln(2) + \frac{\pi}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} - \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{8m^3} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

$$J_2 = -\frac{\pi^2}{32} \ln(2) + \frac{\pi}{4} G + \frac{1}{16} \eta(3) - \frac{1}{2} \eta(3)$$

$$J_2 = -\frac{\pi^2}{32} \ln(2) + \frac{\pi}{4} G - \frac{7}{16} \eta(3)$$

$$J_2 = -\frac{\pi^2}{32} \ln(2) + \frac{\pi}{4} G - \frac{21}{64} \zeta(3)$$

Combining J_1 & J_2 to get J from equation 2

$$J = \frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} - \frac{1}{2} \ln(2) - \frac{\pi^2}{32} \ln(2) + \frac{\pi}{4} G - \frac{21}{64} \zeta(3)$$

Now solving for K

$$K = \int_0^1 x^2 \ln\left(\frac{\ln(x)}{x^3 - 1}\right) dx = \int_0^1 x^2 \ln\left(\frac{-\ln(x)}{1 - x^3}\right) dx$$

$$K = \int_0^1 x^2 \ln(-\ln(x)) dx - \int_0^1 x^2 \ln(1 - x^3) dx$$

Let $K = K_1 - K_2$... equation 3

$$K_1 = \int_0^1 x^2 \ln(-\ln(x)) dx$$

$$x = e^{-t}, dx = -e^{-t} dt, t \in (0, \infty)$$

$$K_1 = \int_0^{\infty} \ln(t) e^{-3t} dt$$

Evaluating via the derivative of the gamma function

$$K_1 = \frac{\partial}{\partial a} \Big|_{a=0} \int_0^{\infty} t^a e^{-3t} dt = \frac{1}{3} \frac{\partial}{\partial a} \frac{\Gamma(a+1)}{3^a} \Big|_{a=0}$$

$$K_1 = \frac{1}{3} \frac{\Gamma(a+1)}{3^a} (\psi(a+1) - \ln(3)) \Big|_{a=0}$$

$$K_1 = \frac{1}{3} (\psi(1) - \ln(3))$$

$$K_1 = -\frac{\gamma}{3} - \frac{\ln(3)}{3}$$

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$$K_2 = \int_0^1 x^2 \ln(1 - x^3) dx$$

$$t = 1 - x^3, dt = -3x^2 dx$$

$$K_2 = \frac{1}{3} \int_0^1 \ln(t) dt = \frac{t}{3} (\ln(t) - 1) \Big|_0^1$$

$$K_2 = -\frac{1}{3}$$

$\therefore$

$$K = \frac{1}{3} - \frac{\gamma}{3} - \frac{\ln(3)}{3}$$

Finally: From equation 1, $I = J + K$

$$I = \frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} - \frac{1}{2} \ln(2) - \frac{\pi^2}{32} \ln(2) + \frac{\pi}{4} G - \frac{21}{64} \zeta(3) + \frac{1}{3} - \frac{\gamma}{3} - \frac{\ln(3)}{3}$$

$$I = \frac{1}{192} (\pi^3 - 12\pi^2 - 6\pi^2 \ln(2) + 48\pi - 96\ln(2) + 48\pi G - 63\zeta(3) - 64\gamma - 64\ln(3) + 64) \blacksquare$$