

Find :

$$\Omega = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 \psi^{(0)} \left(\frac{1}{2} + \frac{1}{2}x \right) dx$$

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Solution 1 by Pham Duc Nam-Vietnam

$$\begin{aligned} \Omega &= \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 \psi^{(0)} \left(\frac{1}{2} + \frac{1}{2}x \right) dx = \int_0^{\frac{1}{2}} x^3 \left(\psi \left(\frac{1}{2} + \frac{1}{2}x \right) - \psi \left(\frac{1}{2} - \frac{1}{2}x \right) \right) dx = \\ &= \pi \int_0^{\frac{1}{2}} x^3 \tan \left(\frac{\pi x}{2} \right) dx \stackrel{t=\frac{\pi x}{2}}{\cong} \frac{16}{\pi^3} \int_0^{\frac{\pi}{4}} t^3 \tan(t) dt \stackrel{IBP/S}{\cong} \\ &= \frac{16}{\pi^3} \left(-[t^3 \ln(\cos(t))] \Big|_0^{\frac{\pi}{4}} + 3 \int_0^{\frac{\pi}{4}} t^2 \ln(\cos(t)) dt \right) = \frac{1}{8} \ln(2) + \frac{48}{\pi^3} \underbrace{\int_0^{\frac{\pi}{4}} t^2 \ln(\cos(t)) dt}_A \\ A &= \int_0^{\frac{\pi}{4}} t^2 \ln(\cos(t)) dt = \int_0^{\frac{\pi}{4}} t^2 \left(-\ln(2) - \sum_{k=1}^{\infty} \frac{(-1)^k \cos(2kt)}{k} \right) dt = \\ &= -\frac{\pi^3}{192} \ln(2) - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \int_0^{\frac{\pi}{4}} t^2 \cos(2kt) dt = \\ &= -\frac{\pi^3}{192} \ln(2) - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(-\frac{1}{4} \frac{1}{k^3} \sin \left(\frac{\pi k}{2} \right) + \frac{\pi}{8} \frac{1}{k^2} \cos \left(\frac{\pi k}{2} \right) + \frac{\pi^2}{32} \frac{1}{k} \sin \left(\frac{\pi k}{2} \right) \right) \\ &\therefore \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4} \sin \left(\frac{\pi k}{2} \right) = - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4} = -\beta(4) \\ &\therefore \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \cos \left(\frac{\pi k}{2} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)^3} = -\frac{3}{32} \zeta(3) \\ &\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin \left(\frac{\pi k}{2} \right) = - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = -G \end{aligned}$$

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$$\begin{aligned}
 A &= -\frac{\pi^3}{192} \ln(2) + \frac{1}{4}(-\beta(4)) - \frac{\pi}{8} \left(-\frac{3}{32} \zeta(3) \right) - \frac{\pi^2}{32}(-G) = \\
 &\quad -\frac{1}{4}\beta(4) + \frac{3\pi}{256}\zeta(3) + \frac{\pi^2}{32}G - \frac{\pi^3}{192}\ln(2) \\
 \Omega &= \frac{1}{8}\ln(2) + \frac{48}{\pi^3} \left(-\frac{1}{4}\beta(4) + \frac{3\pi}{256}\zeta(3) + \frac{\pi^2}{32}G - \frac{\pi^3}{192}\ln(2) \right) \\
 \Omega &= -\frac{12}{\pi^3}\beta(4) + \frac{9}{16\pi^2}\zeta(3) + \frac{3G}{2\pi} - \frac{1}{8}\ln(2)
 \end{aligned}$$

Solution 2 by Yang Silva Cartolin-Peru

$$\Omega = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 \psi^{(0)}\left(\frac{1}{2} + \frac{x}{2}\right) dx \xrightarrow{\text{We do: } x \rightarrow -x} \Omega = - \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 \psi^{(0)}\left(\frac{1}{2} - \frac{x}{2}\right) dx$$

$$\text{Thus: } 2\Omega = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 \left[\psi^{(0)}\left(\frac{1}{2} + \frac{x}{2}\right) - \psi^{(0)}\left(\frac{1}{2} - \frac{x}{2}\right) \right] dx$$

We know that:

$$\begin{aligned}
 \psi^{(0)}(z) - \psi^{(0)}(1-z) &= -\pi \cot(\pi z) \\
 \Rightarrow 2\Omega &= \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 \cot\left(\pi \left(\frac{1}{2} + \frac{x}{2}\right)\right) dx = \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 \tan\left(\frac{\pi x}{2}\right) dx
 \end{aligned}$$

$$\xrightarrow{\text{Even Function}} 2\Omega = 2\pi \int_0^{\frac{1}{2}} x^3 \tan\left(\frac{\pi x}{2}\right) dx$$

$$\Omega = \pi \int_0^{\frac{1}{2}} x^3 \tan\left(\frac{\pi x}{2}\right) dx \xrightarrow{\text{We do: } u = \frac{\pi x}{2}} \Omega = \frac{16}{\pi^3} \int_0^{\frac{\pi}{4}} u^3 \tan(u) du$$

$$\text{I.B.P: } \begin{cases} a = u^3 \rightarrow da = 3u^2 du \\ db = \tan(u) du \rightarrow b = -\log(\cos(u)) \end{cases}$$

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$$\Rightarrow \Omega = \frac{16}{\pi^3} \left([-u^3 \log(\cos(u))]_0^{\frac{\pi}{4}} + 3 \int_0^{\frac{\pi}{4}} u^2 \log(\cos(u)) du \right)$$

We know that: *Fourier Series*

$$\log(\cos(x)) = -\log(2) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cos(2kx)$$

$$\Rightarrow \Omega = \frac{16}{\pi^3} \left(\frac{\pi^3}{128} \log(2) + 3 \int_0^{\frac{\pi}{4}} u^2 \left(-\log(2) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cos(2ku) \right) du \right)$$

$$\Omega = \frac{1}{8} \log(2) + \frac{48}{\pi^3} \left(-\log(2) \int_0^{\frac{\pi}{4}} u^2 du - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \int_0^{\frac{\pi}{4}} u^2 \cos(2ku) du \right)$$

$$\Omega = \frac{1}{8} \log(2) + \frac{48}{\pi^3} \left(-\frac{\pi^3}{192} \log(2) \right) - \frac{48}{\pi^3} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{\pi^2}{32k} \sin\left(\frac{\pi k}{2}\right) + \frac{\pi}{8k^2} \cos\left(\frac{\pi k}{2}\right) - \frac{1}{4k^3} \sin\left(\frac{\pi k}{2}\right) \right)$$

$$\Omega = -\frac{1}{8} \log(2) - \frac{48}{\pi^3} \left(\frac{\pi^2}{32} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin\left(\frac{\pi k}{2}\right) + \frac{\pi}{8} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \cos\left(\frac{\pi k}{2}\right) - \frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4} \sin\left(\frac{\pi k}{2}\right) \right)$$

$$\text{Let: } \blacksquare S_1 = \frac{\pi^2}{32} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sin\left(\frac{\pi k}{2}\right) - \frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4} \sin\left(\frac{\pi k}{2}\right)$$

$$\xrightarrow{k=2n+1} S_1 = \frac{\pi^2}{32} \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)^2} (-1)^n - \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)^4} (-1)^n$$

$$S_1 = -\frac{\pi^2}{32} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4} \therefore$$

$$\blacksquare S_2 = \frac{\pi}{8} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \cos\left(\frac{\pi k}{2}\right) \xrightarrow{k=2n} S_2 = \frac{\pi}{8} \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{(2n)^3} (-1)^n$$

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$$S_2 = -\frac{\pi}{64} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \therefore$$

$$\Rightarrow \Omega = -\frac{1}{8} \log(2) + \frac{3}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} + \frac{3}{4\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} - \frac{12}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4}$$

We know that: ■ **Dirichlet Beta Function:** $\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \beta(2) = G \wedge \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4} = \beta(4)$$

■ **Dirichlet Beta Function:** $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \eta(3) = \frac{3}{4}\zeta(3)$$

Therefore: $\Omega = -\frac{1}{8} \log(2) + \frac{3G}{2\pi} + \frac{9\zeta(3)}{16\pi^2} - \frac{12\beta(4)}{\pi^3}$

$$\Omega = -\frac{1}{8} \log(2) + \frac{8}{9\pi^3} \left(\frac{27\pi^2}{16} G + \frac{81\pi}{128} \zeta(3) - \frac{27}{2} \beta(4) \right) \therefore$$

Solution 3 by Omkumar Rao-India

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 \psi^{(0)}\left(\frac{1}{2} + \frac{1}{2}x\right) dx \stackrel{\frac{1}{2} + \frac{1}{2}x \rightarrow x}{\cong} 2 \int_{\frac{1}{4}}^{\frac{3}{4}} (2x-1)^3 \psi(x) dx \stackrel{x \rightarrow 1-x}{\cong} -2 \int_{\frac{1}{4}}^{\frac{3}{4}} (2x-1)^3 \psi(1-x) dx$$

$$2I = \int_{\frac{1}{4}}^{\frac{3}{4}} 2(2x-1)^3 [\psi(x) - \psi(1-x)] dx = -2\pi \int_{\frac{1}{4}}^{\frac{3}{4}} (2x-1)^3 \cot(\pi x) dx \stackrel{2x-1 \rightarrow 2x}{\cong} 16\pi \int_{-\frac{1}{4}}^{\frac{1}{4}} x^3 \tan(\pi x) dx$$

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$$I = 16\pi \int_0^{\frac{1}{4}} x^3 \tan(\pi x) dx \stackrel{\pi x \rightarrow x}{=} \frac{16}{\pi^3} \int_0^{\frac{\pi}{4}} x^3 \tan(x) dx \stackrel{\text{By parts}}{=} \frac{16}{\pi^3} \left(\frac{\pi^3}{128} \ln 2 \right.$$

$$\left. + 3 \int_0^{\frac{\pi}{4}} x^2 \frac{\ln(\cos(x))}{-\ln 2 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \cos(2mx)}{m}} dx \right)$$

$$I = \frac{16}{\pi^3} \left(\frac{\pi^3}{128} \ln 2 + 3 \int_0^{\frac{\pi}{4}} x^2 \left(-\ln 2 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \cos(2mx)}{m} \right) dx \right)$$

$$= \frac{16}{\pi^3} \left(\frac{\pi^3}{128} \ln 2 + 3 \left(-\frac{\pi^3}{192} \ln 2 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \int_0^{\frac{\pi}{4}} x^2 \cos(2mx) dx \right) \right)$$

$$I = \frac{16}{\pi^3} \left(\frac{\pi^3}{128} \ln 2 + 3 \left(-\frac{\pi^3}{192} \ln 2 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\frac{\pi^2}{32m} \sin\left(\frac{m\pi}{2}\right) + \frac{\pi}{8m^2} \cos\left(\frac{m\pi}{2}\right) - \frac{1}{4m^3} \sin\left(\frac{m\pi}{2}\right) \right) \right) \right)$$

$$I = \frac{16}{\pi^3} \left(\frac{\pi^3}{128} \ln 2 \right.$$

$$\left. + 3 \left(-\frac{\pi^3}{192} \ln 2 + \underbrace{\sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1} \pi^2}{32m^2} \sin\left(\frac{m\pi}{2}\right) \right)}_{\frac{\pi^2}{32} G} + \underbrace{\sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1} \pi}{8m^3} \cos\left(\frac{m\pi}{2}\right) \right)}_{\frac{3\pi}{256} \zeta(3)} - \underbrace{\sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1}}{4m^4} \sin\left(\frac{m\pi}{2}\right) \right)}_{\frac{1}{4} \beta(4)} \right) \right)$$

$$I = \frac{16}{\pi^3} \left(\frac{\pi^3}{128} \ln 2 + 3 \left(-\frac{\pi^3}{192} \ln 2 + \frac{\pi^2}{32} G + \frac{3\pi}{256} \zeta(3) - \frac{1}{4} \beta(4) \right) \right)$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 \psi^{(0)}\left(\frac{1}{2} + \frac{1}{2}x\right) dx = \frac{3}{2\pi} G + \frac{9}{16\pi^2} \zeta(3) - \frac{1}{8} \ln 2 - \frac{12}{\pi^3} \beta(4)$$