

Find :

$$\Delta = \int_0^{\frac{\pi}{2}} x^2 \ln \left(\frac{1 + \cos(x)}{\sqrt{x}} \right) dx$$

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Solution 1 by Nawar Alasadi-Iraq

$$\Delta = \int_0^{\frac{\pi}{2}} x^2 \ln \left(\frac{1 + \cos(x)}{\sqrt{x}} \right) dx = I_1 - I_2$$

$$I_2 = \frac{1}{2} \int_0^{\frac{\pi}{2}} x^2 \ln(x) dx = \frac{1}{2} \left[\frac{x^3}{3} \ln(x) - \frac{x^3}{9} \right]_0^{\frac{\pi}{2}} = \frac{\pi^3}{48} \ln\left(\frac{\pi}{2}\right) - \frac{\pi^3}{144}$$

$$I_1 = \int_0^{\frac{\pi}{2}} x^2 \ln(1 + \cos(x)) dx = \int_0^{\frac{\pi}{2}} x^2 \ln \left(2 \cos^2 \left(\frac{x}{2} \right) \right) dx =$$

$$\ln(2) \int_0^{\frac{\pi}{2}} x^2 dx + 2 \int_0^{\frac{\pi}{2}} x^2 \ln \left(2 \cos \left(\frac{x}{2} \right) \right) dx \stackrel{(t=\frac{x}{2})}{=} \frac{\pi^3}{24} \ln(2) + 16 \int_0^{\frac{\pi}{4}} t^2 \ln(\cos(t)) dt$$

$$\therefore \ln(\cos(t)) = -\ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos(2nt)$$

$$\int_0^{\frac{\pi}{4}} t^2 \ln(\cos(t)) dt = \frac{\pi^3}{192} \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \underbrace{\int_0^{\frac{\pi}{4}} t^2 \cos(2nt) dt}_{J_n}$$

$$J_n = \left[t^3 \frac{\sin(2nt)}{2n} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2t \frac{\sin(2nt)}{2n} dt =$$

$$\frac{\pi^2}{32n} \sin\left(\frac{n\pi}{2}\right) + \left[t \frac{\cos(2nt)}{2n^2} \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{\cos(2nt)}{2n^2} dt =$$

$$\left(\frac{\pi^2}{32n} - \frac{1}{4n^3} \right) \sin\left(\frac{n\pi}{2}\right) + \frac{\pi}{8n^2} \cos\left(\frac{n\pi}{2}\right)$$

$$I_1 = -\frac{\pi^3}{24} \ln(2) + 16 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} J_n = -\frac{\pi^3}{24} \ln(2) + 16(S_{\text{even}} + S_{\text{old}})$$

$$S_{even}(n = 2k) = \sum_{k=1}^{\infty} -\frac{1}{2k} \left(\frac{\pi}{8(2k)^2} (-1)^k \right) = \frac{\pi}{64} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} = \frac{3\pi}{256} \zeta(3)$$

$$S_{old}(n = 2k - 1) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \left(\frac{\pi^2}{32(2k-1)} - \frac{1}{4(2k-1)^3} \right) =$$

$$\frac{\pi^2}{32} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^4} = \frac{\pi^2}{32} G - \frac{1}{4} \beta(4)$$

$$I_1 = -\frac{\pi^3}{24} \ln(2) + 16(S_{even} + S_{old}) =$$

$$-\frac{\pi^3}{24} \ln(2) + 16 \left(\frac{3\pi}{256} \zeta(3) + \frac{\pi^2}{32} G - \frac{1}{4} \beta(4) \right) = -\frac{\pi^3}{24} \ln(2) + \frac{3\pi}{16} \zeta(3) + \frac{\pi^2}{2} G - 4\beta(4)$$

$$\Delta = I_1 - I_2 = \frac{\pi^3}{24} \ln(2) + \frac{3\pi}{16} \zeta(3) + \frac{\pi^2}{2} G - 4\beta(4) - \frac{\pi^3}{48} \ln\left(\frac{\pi}{2}\right) + \frac{\pi^3}{144} =$$

$$\frac{3\pi}{16} \zeta(3) + \frac{\pi^2}{2} G - 4\beta(4) - \frac{\pi^3}{48} \ln(2\pi) + \frac{\pi^3}{144}$$

Solution 2 by Yang Silva Cartolin-Peru

$$I = \int_0^{\frac{\pi}{2}} x^2 \log(1 + \cos(x)) dx - \frac{1}{2} \int_0^{\frac{\pi}{2}} x^2 \log(x) dx$$

We know that: $1 + \cos(x) = 2\cos^2\left(\frac{x}{2}\right)$

$$I = \log(2) \int_0^{\frac{\pi}{2}} x^2 dx + 2 \int_0^{\frac{\pi}{2}} x^2 \log\left(\cos\left(\frac{x}{2}\right)\right) dx - \frac{1}{2} \int_0^{\frac{\pi}{2}} x^2 \log(x) dx$$

$$I = \frac{\pi^3}{24} \log(2) + 2i_1 - \frac{1}{2}i_2$$

$$\blacksquare i_2 = \int_0^{\frac{\pi}{2}} x^2 \log(x) dx \xrightarrow{I.B.P} i_2 = \left[\frac{x^3}{3} \log(x) \right]_0^{\frac{\pi}{2}} - \frac{1}{3} \int_0^{\frac{\pi}{2}} x^2 dx = \frac{\pi^3}{24} \log\left(\frac{\pi}{2}\right) - \frac{\pi^3}{72} \therefore$$

$$\blacksquare i_1 = \int_0^{\frac{\pi}{2}} x^2 \log\left(\cos\left(\frac{x}{2}\right)\right) dx$$

We know that: $\cos\left(\frac{x}{2}\right) = \frac{e^{\frac{ix}{2}} + e^{-\frac{ix}{2}}}{2} \Rightarrow 2e^{\frac{ix}{2}} \cos\left(\frac{x}{2}\right) = 1 + e^{ix}$

$$\log(2) + \log\left(e^{\frac{ix}{2}}\right) + \log\left(\cos\left(\frac{x}{2}\right)\right) = \log(1 + e^{ix})$$

$$\log\left(\cos\left(\frac{x}{2}\right)\right) = \log(1 + e^{ix}) - \log(2) - \frac{ix}{2}$$

$$\Re\left\{\log\left(\cos\left(\frac{x}{2}\right)\right)\right\} = \Re\{\log(1 + e^{ix})\} - \log(2)$$

$$\Rightarrow i_1 = \Re\left\{\int_0^{\frac{\pi}{2}} x^2 \log(1 + e^{ix}) dx\right\} - \log(2) \int_0^{\frac{\pi}{2}} x^2 dx$$

$$i_1 = \Re\left\{\int_0^{\frac{\pi}{2}} x^2 \log(1 + e^{ix}) dx\right\} - \frac{\pi^3}{24} \log(2)$$

We know that: $\log(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n \xrightarrow{z=e^{ix}} \log(1 + e^{ix}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{inx}$

$$\Re\{\log(1 + e^{ix})\} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos(nx)$$

$$\Rightarrow i_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^{\frac{\pi}{2}} x^2 \cos(nx) dx - \frac{\pi^3}{24} \log(2)$$

$$i_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\frac{\pi^2}{4n} \sin\left(\frac{\pi n}{2}\right) - \frac{2}{n^3} \sin\left(\frac{\pi n}{2}\right) + \frac{\pi}{n^2} \cos\left(\frac{\pi n}{2}\right) \right] - \frac{\pi^3}{24} \log(2)$$

$$i_1 = -\frac{\pi^2}{4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin\left(\frac{\pi n}{2}\right) + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \sin\left(\frac{\pi n}{2}\right) - \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \cos\left(\frac{\pi n}{2}\right) - \frac{\pi^3}{24} \log(2)$$

Let:

$$\gg S_1 = -\frac{\pi^2}{4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin\left(\frac{\pi n}{2}\right) + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \sin\left(\frac{\pi n}{2}\right)$$

$$\xrightarrow{n=2k+1} S_1 = \frac{\pi^2}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} - 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^4} \therefore$$

$$\gg S_2 = -\pi \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \cos\left(\frac{\pi n}{2}\right) \xrightarrow{n=2k} S_2 = \frac{\pi}{8} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3}$$

We know that: ■ **Dirichlet Beta Function:** $\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \beta(2) = G \wedge \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^4} = \beta(4)$$

■ **Dirichlet Beta Function:** $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \eta(3) = \frac{3}{4}\zeta(3)$$

$$\Rightarrow i_1 = \frac{\pi^2}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} - 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^4} + \frac{\pi}{8} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} - \frac{\pi^3}{24} \log(2)$$

$$i_1 = \frac{\pi^2}{4} G - 2\beta(4) + \frac{3\pi}{32}\zeta(3) - \frac{\pi^3}{24} \log(2) \therefore$$

$$I = \frac{\pi^3}{24} \log(2) + 2 \left(\frac{\pi^2}{4} G - 2\beta(4) + \frac{3\pi}{32}\zeta(3) - \frac{\pi^3}{24} \log(2) \right) + \frac{1}{2} \left(\frac{\pi^3}{24} \log\left(\frac{\pi}{2}\right) - \frac{\pi^3}{72} \right)$$

$$I = -\frac{\pi^3}{48} \log(2\pi) + \frac{\pi^2}{2} G - 4\beta(4) + \frac{3\pi}{16}\zeta(3) + \frac{\pi^3}{144} \therefore$$

Solution 3 by Omkumar Rao-India

$$K = \underbrace{\ln 2 \int_0^{\frac{\pi}{2}} x^2 dx}_{\frac{\pi^3}{24} \ln 2} - \underbrace{\frac{1}{2} \int_0^{\frac{\pi}{2}} x^2 \ln x dx}_{\frac{\pi^3}{48} \ln\left(\frac{\pi}{2}\right) - \frac{\pi^3}{144}} + \underbrace{2 \int_0^{\frac{\pi}{2}} x^2 \ln\left(\cos\left(\frac{x}{2}\right)\right) dx}_M$$

$$K = \frac{\pi^3}{24} \ln 2 - \frac{\pi^3}{48} \ln\left(\frac{\pi}{2}\right) + \frac{\pi^3}{144} + M$$

$$M = 2 \int_0^{\frac{\pi}{2}} x^2 \ln \left(\cos \left(\frac{x}{2} \right) \right) dx \stackrel{x \rightarrow 2x}{\cong} 16 \int_0^{\frac{\pi}{4}} x^2 \underbrace{\frac{\ln(\cos(x))}{-\ln 2 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \cos(2mx)}}_{\substack{\ln(\cos(x)) \\ -\ln 2 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \cos(2mx)}} dx$$

$$= 16 \int_0^{\frac{\pi}{4}} x^2 \left(-\ln 2 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \cos(2mx) \right) dx$$

$$M = -\frac{\pi^3}{12} \ln 2 + 16 \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\int_0^{\frac{\pi}{4}} x^2 \cos(2mx) dx \right)$$

$$M = -\frac{\pi^3}{12} \ln 2 + 16 \left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\frac{\pi^2}{32m} \sin \left(\frac{m\pi}{2} \right) + \frac{\pi}{8m^2} \cos \left(\frac{m\pi}{2} \right) - \frac{1}{4m^3} \sin \left(\frac{m\pi}{2} \right) \right) \right)$$

$$M = -\frac{\pi^3}{12} \ln 2 + 16 \left(\sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1} \pi^2}{32m^2} \sin \left(\frac{m\pi}{2} \right) \right) + \sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1} \pi}{8m^3} \cos \left(\frac{m\pi}{2} \right) \right) - \sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1}}{4m^4} \sin \left(\frac{m\pi}{2} \right) \right) \right)$$

$$\sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1} \pi^2}{32m^2} \sin \left(\frac{m\pi}{2} \right) \right) = \frac{\pi^2}{32} \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{(2p-1)^2} = \frac{\pi^2}{32} G$$

$$\sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1} \pi}{8m^3} \cos \left(\frac{m\pi}{2} \right) \right) = \frac{\pi}{64} \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p^3} = \frac{3\pi}{256} \zeta(3)$$

$$\sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1}}{4m^4} \sin \left(\frac{m\pi}{2} \right) \right) = \frac{1}{4} \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{(2p-1)^4} = \frac{1}{4} \beta(4)$$

$$M = -\frac{\pi^3}{12} \ln 2 + 16 \left(\frac{\pi^2}{32} G + \frac{3\pi}{256} \zeta(3) - \frac{1}{4} \beta(4) \right)$$

$$K = \frac{\pi^3}{24} \ln 2 - \frac{\pi^3}{48} \ln \left(\frac{\pi}{2} \right) + \frac{\pi^3}{144} - \frac{\pi^3}{12} \ln 2 + 16 \left(\frac{\pi^2}{32} G + \frac{3\pi}{256} \zeta(3) - \frac{1}{4} \beta(4) \right)$$

$$K = \frac{\pi^3}{144} + \frac{\pi^2}{2} G + \frac{3\pi}{16} \zeta(3) - 4\beta(4) - \frac{\pi^3}{48} \ln(2\pi)$$